Math 192r, Problem Set #7 (Solutions)

1. (a) *How many different polygonal paths of length $n$ are there that start at the point $(0,0)$ and then take $n$ steps of length 1, such that each step is either rightward, leftward, or upward, and such that no point gets visited more than once? Give an explicit formula.*

   This is the same as the number of strings of length $n$ consisting of the symbols $R$, $L$, and $U$ (short for Right, Left, and Up, respectively) such that no $R$ is followed by an $L$ and no $L$ is followed by an $R$. The associated 1-step transfer matrix

   $$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

   has characteristic polynomial $(t-1)(t^2-2t-1)$, so the answer is of the form $A+Br^n+Cs^n$ where $r=1+\sqrt{2}$, $s=1-\sqrt{2}$, and $A,B,C$ are undetermined coefficients. Using the fact that the number of polygonal paths of the desired kind equals 1, 3, and 7 when $n$ is 0, 1, and 2, respectively, we get $A=0$, $B=1+\sqrt{2}$, and $C=1-\sqrt{2}$, so that the final answer is $\frac{1}{2}((1+\sqrt{2})^{n+1}+(1-\sqrt{2})^{n+1})$.

   To do this in Maple, one might proceed as follows:

```
with(linalg):
m := matrix(3,3,[1,0,1,0,1,1,1,1,1]);
p := charpoly(m,t);
sols := solve(p=0,t);
r := sols[2];
s := sols[3];
ans:=solve({A+B*r+C*s=3,A+B*r^2+C*s^2=7,
        A+B*r^3+C*s^3=17},{A,B,C});
```

   The result is a set whose three elements are equations giving the values of $A$, $C$, and $B$ respectively. (Note that the Maple command I used doesn't return the values of the variables in the same order as I specified them! Does anyone know of a variant of my command that doesn't suffer from this defect?) By the way, the command `with(linalg)` only needs to be done once per session.

(b) *If one chooses at random one of the paths of length n described in part (a) (so that each of the length-n paths has an equal chance of being chosen), what is the expected value of the y-coordinate of the last point on the path? Find a constant c so that this expected value is asymptotic to cn.*

An appropriate generating function is $\sum_{n\geq 1}(uM^n v)x^n$, where $u = (1, 1, y)$, $v = (1, 1, 1)^T$ (the transpose), and $M$ is a modified version of the preceding transition matrix in which the 1's that correspond to Up-steps are replaced by $y$'s, so that when we multiply the matrix by itself, obtaining a matrix of polynomials in $y$, a term equal to $y^k$ corresponds to a path that takes $k$ Up-steps. (After we've expressed this generating function in closed form, we'll be able to differentiate it to get at the information we seek.) The entry $y$ in the vector $u$ occurs because it corresponds to taking a step in the Up direction. The generating function can be written as the sum of the nine entries of $u(\sum(Mx)^n)v = u(I - Mx)^{-1}v$, where the matrix $Mx$ is

$$\begin{pmatrix} x & 0 & xy \\ 0 & x & xy \\ x & x & xy \end{pmatrix}.$$

We can use Maple for this:

```
with(linalg):
Mx := [[x,0,x*y],[0,x,x*y],[x,x,x*y]];
Id := [[1,0,0],[0,1,0],[0,0,1]];
u := [[x,x,x*y]];
v := [[1],[1],[1]];
inv := inverse(Id-Mx);
ans := simplify(multiply(u,inv,v)[1,1]);
```

(Note that for Maple, $xy$ must be written as `x*y`; also note that `Mx` is just an indivisible symbol. Observe that the symbol `I` is reserved for the square root of minus 1. Finally, note that the output of `multiply(u,inv,v)` is a 1-by-1 matrix, not a number; hence the need to extract its 1,1 element with the matrix-entry-extraction operator `[1,1]`.) The answer `ans` turns out to be the simple expression

$$\frac{2+y+xy}{1-x-xy-x^2y}.$$

If we differentiate this with respect to $y$ and then set $y=1$, we will obtain the generating function in which the coefficient of $x^n$ is the sum of the heights of all the polygonal paths.

```
heights := simplify(subs(y=1,diff(ans,y)));
```

gives

$$\frac{x(1+x)^2}{(1-2x-x^2)^2}.$$

To find the asymptotic behavior of the coefficients of this generating function, use partial fractions over the field generated over the rationals by the square root of 2:

```
convert(heights, parfrac, x, sqrt(2));
```

Unfortunately, Maple gives us an answer in which the denominators of the four terms are of the form $x+a$ instead of $1+bx$, but this is only a minor annoyance. The term that controls the growth rate is the term whose denominator is quadratic and vanishes closest to $x=0$. This is the term

$$\frac{1}{4}\frac{-1+\sqrt{2}}{(x+1-\sqrt{2})^2}=\frac{1+\sqrt{2}}{4}(1-x(1+\sqrt{2}))^{-2}.$$

Now we may apply the binomial theorem with exponent $-2$: the coefficient of $x^n$ in the preceding generating function equals

$$\frac{1+\sqrt{2}}{4}\binom{-2}{n}(-(1+\sqrt{2}))^n=\frac{1+\sqrt{2}}{4}\binom{n+1}{n}(1+\sqrt{2})^n,$$

which grows like $\frac{n}{4}(1+\sqrt{2})^{n+1}$. The answer to part (a) grows like $\frac{1}{2}(1+\sqrt{2})^{n+1}$, so, taking the ratio, we find that the expected height tends to the limit $n/2$. (There must be a nice way to see this!)

2. *Look at the (infinite-dimensional) space of self-reciprocal Laurent polynomials in $t$, that is, Laurent polynomials in $t$ that are unaffected by the substitution $t \to 1/t$. The space has two natural bases: $x_j = t^j + 1/t^j$ (with $j \geq 0$) and $y_k = (t+1/t)^k$ (with $k \geq 0$).*

   (a) *Give an explicit formula for the entries of the (triangular) matrix that expresses the $y_k$ in terms of the $x_j$.*

   I'll give the matrices in upper-triangular form (though I see that in the statement of the problem I used lower-triangular form, since I asserted in part (b) that each *row* has only finitely many non-zero entries). For $j = 0$, the $j,k$th entry in the matrix (the coefficient of $x_j$ in the expansion of $y_k$) is $\frac{1}{2}\binom{k}{k/2}$ (which we interpret as 0 if $k/2$ is non-integer). For $j > 0$, the $j,k$th entry is simply $\binom{k}{(j+k)/2}$ (which we interpret as 0 if $k < j$ or if $(j+k)/2$ is not an integer). Here's the excerpt of the matrix with $j,k$ running between 0 and 9:

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 3 & 0 & 10 & 0 & 35 & 0 \\ 0 & 1 & 0 & 3 & 0 & 10 & 0 & 35 & 0 & 126 \\ 0 & 0 & 1 & 0 & 4 & 0 & 15 & 0 & 56 & 0 \\ 0 & 0 & 0 & 1 & 0 & 5 & 0 & 21 & 0 & 84 \\ 0 & 0 & 0 & 0 & 1 & 0 & 6 & 0 & 28 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 7 & 0 & 36 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

   One way to make this array is as follows:

```
entry := proc(j,k)
        if (j=0 and k mod 2 = 0) then
                simplify((1/2)*binomial(k,k/2));
        elif (k>=j and j+k mod 2 = 0) then
                binomial(k,(j+k)/2);
        else 0; fi; end;
m := matrix(10,10,[seq(seq(entry(j,k),k=0..9),j=0..9)]);
```

   (b) *Look at the inverse of this matrix (which expresses the $x_j$ in terms of the $y_k$). Try to conjecture a formula for the entries, or if you* *can't get that far, look for patterns and conjecture some interesting properties. (E.g., in each row of the infinite matrix, there are only finitely many non-zero values. What is their sum? What is the sum of their absolute values?) For full credit, you will need to conjecture (though not necessarily prove) a formula for all the entries.*

In Maple (with the `linalg` library loaded), `inverse(m)` gives

$$\begin{pmatrix} 2 & 0 & -2 & 0 & 2 & 0 & -2 & 0 & 2 & 0 \\ 0 & 1 & 0 & -3 & 0 & 5 & 0 & -7 & 0 & 9 \\ 0 & 0 & 1 & 0 & -4 & 0 & 9 & 0 & -16 & 0 \\ 0 & 0 & 0 & 1 & 0 & -5 & 0 & 14 & 0 & -30 \\ 0 & 0 & 0 & 0 & 1 & 0 & -6 & 0 & 20 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -7 & 0 & 27 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Column-sums are $2, 1, -1, -2, -1, 1, 2, 1, -1, -2, \ldots$, which seems to be a periodic sequence. If we take absolute values before summing, we get $2, 1, 3, 4, 7, 11, 18, 29, 47, 76, \ldots$, which looks like the Lucas sequence. We can certainly conjecture that the $k, j$ entry has sign $(-1)^{(k-j)/2}$. (We index rows by $k$ and columns by $j$ now, since this is the inverse matrix.) One might also note that the successive non-zero diagonals (leaving aside the principal diagonal, which has a glitch in its first entry) appear to be given by polynomials of degree 1, 2, 3, etc.

The key to finding a formula for the entries lies in the observation that most of the entries in the $j$ column are divisible by $j$, or nearly so. In fact, if you throw out the 0 row and 0 column, get rid of all the signs, and divide each entry by its column-index and then multiply each entry by its row-index, you get the integer matrix

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 & 3 & 0 & 4 & 0 \\ 0 & 0 & 1 & 0 & 3 & 0 & 5 & 0 & 10 \\ 0 & 0 & 0 & 1 & 0 & 4 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 5 & 0 & 15 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

which is just Pascal's triangle! So the coefficient of $y_k$ in the expansion of $x_j$ (relative to the basis $y_0, y_1, \ldots$) is

$$(-1)^{(k-j)/2} \frac{j}{k} \binom{(j+k-2)/2}{(j-k)/2}$$

as long as $j \geq k \geq 1$ and $j, k$ have the same parity; in the case $k = 0$, the coefficient is

$$(-1)^{(j-k)/2}\, 2$$

as long as $j \geq 0$ and $j, k$ have the same parity; otherwise, the coefficient vanishes.