Math 192r, Problem Set #6: Solutions

1. *For each even integer $n \geq 2$, we can represent each domino tiling of a 3-by-n rectangle by a codeword $(a_1, a_2, \ldots, a_n)$, where a_k is the number of vertical dominos in the kth column (always either 0 or 1). Note that two different tilings can have the same codeword; e.g., for $n = 2$ there are three tilings but only two codewords (namely $(0,0)$ and $(1,1)$). Formulate a conjecture for the number of codewords that occur for general n.*

   Here is some Maple code that does quite a lot:

```
with(linalg):
m :=proc(k) matrix(3,3,
        [1+2*x(k),x(k),x(k),y(k),1,0,y(k),0,1]); end;
count := proc(n) nops(expand(multiply(
        seq(m(i),i=1..n))[1,1])); end;
```

   The first line tells Maple to use the linear algebra package `linalg`. The second line tells Maple to define `m`(n) as the 3-by-3 matrix

$$\begin{pmatrix} 1+2x(k) & x(k) & x(k) \\ y(k) & 1 & 0 \\ y(k) & 0 & 1 \end{pmatrix}.$$

   The third line tells Maple that when it receives the command `count`(n), it should take the product of n such 3-by-3 matrices, look at the upper left entry in the product, expand it out, and count the number of operands (that is, count the number of terms in the expansion).

   This calls out for some explanation. Suppose we assign weight $x(k)$ (resp. $y(k)$) to either of the two vertical dominos occurring in the $2k-1$st (resp. $2k$th) column of a 3-by-n rectangle, and suppose we assign to each tiling of this rectangle a weight equal to the products of the weights of the constituent vertical tiles. Then it is clear that the weight of a tiling determines its codeword, and vice versa, with the presence or absence of the weight $x(k)$ or $y(k)$ indicating the occurrence of a 1 or a 0 at the corresponding location in the codeword. So we can count the distinct codewords by counting the number of terms in the polynomial that is obtained by summing the weights of all the tilings.

We can do this using a 3-by-3 transfer matrix to handle three mutually recursive sequences simultaneously: the first counts domino tilings of a 3-by-$2n$ rectangle, the second counts domino tilings of a 3-by-$2n+1$ rectangle with a bite taken out of the upper right corner, and the third counts domino tilings of a 3-by-$2n+1$ rectangle with a bite taken out of the lower right corner, (There is some redundancy here between the second and third sequence, but it's simpler conceptually, if not computationally, to keep it around.)

Another approach is to generate all the tilings, determine their codewords, and strip out redundancies, using the fact that when Maple takes a union of two sets, it takes care of removing duplicates for you (provided you've represented objects in such a way that Maple's notion of equality coincides with yours).

Under either approach, one finds that the number of codes of domino tilings of a 3-by-$2n$ rectangle appears to equal the number of domino tilings of a 2-by-$2n$ rectangle. It should also be noted that all the polynomials obtained from the Maple code given above have coefficients that are powers of 2. That is, for each codeword, the number of tilings of the 3-by-$2n$ rectangle that have that particular codeword always seems to be a power of 2.

2. *Let a_n be the number of domino tilings of a 4-by-n rectangle, with $n \geq 0$ (we put $a_0 = 1$ by convention).*

   (a) *Prove that the sequence $a_0, a_1, \ldots$ satisfies a linear recurrence relation of order 16 or less.*

   For each vertical line through the tiling that divides one column of height 4 from the next, there are $2^4 = 16$ different ways in which one might see 0 to 4 horizontal dominos being pierced by the line. If we give each of these 16 patterns a symbol, then every domino tiling of a 4-by-n rectangle is represented by a string of $n+1$ symbols from an alphabet of size 16, beginning and ending with the symbol that means "no horizontal dominos pierced by this line" (corresponding to the left end and right end of the rectangle);

call this the first symbol. Conversely, such a codeword corresponds to exactly one tiling if certain forbidden adjancies do not occur, and otherwise corresponds to no tilings at all. Thus the number of (domino-)tilings of the 4-by-n rectangle is equal to the upper-left entry of the nth power of the 16-by-16 matrix whose i, jth entry tells whether the ith symbol can occur next to the jth symbol. By the Cayley-Hamilton theorem, the sequence consisting of these entries must satisfy the 16th-order linear recurrence relation given by the characteristic polynomial of the matrix.

(b) Prove that the sequence $a_0, a_1, \ldots$ satisfies a linear recurrence relation of order 8 or less.

To prove this claim, it's enough to show that half of the 16 symbols cannot occur in such a codeword (so that the 16-by-16 matrix can be replaced by an 8-by-8 matrix). To see this, note that half of the symbols indicate the presence of an odd number of horizontal dominos being pierced by a particular vertical line. That is, if you divide the rectangle in half along the vertical line and remove the horizontal dominos that the line pierces, one is left with two sub-regions, each with odd area. But then these two sub-regions cannot be tiled by dominos (since a domino has even area). Hence only 8 of the 16 symbols can occur in codewords of the form $(1, \ldots, 1)$, and the claim is proved.

(c) Prove that the sequence $a_0, a_1, \ldots$ satisfies a linear recurrence relation of order 6 or less.

Now we show that only 6 of the original 16 symbols can actually participate in a codeword of the form $(1, \ldots, 1)$. Color the constituent squares of the rectangle black and white. If we have a tiling of the rectangle and we cut it along a vertical line (discarding the bisected pieces), the two remaining regions must be tilable by dominos; so each must contain as many black squares as it contains white squares. On the other hand, suppose we bisect the rectangle along the same line and *don't* discard the bisected pieces; then (since each column has an even number of squares) it's clear that each side must contain as many black squares as white. It follows that, among the bisected dominos, there must be as many with a black square on the left and a white square on the right as there are with a white square on the left and a black square on the right. This leaves only six possibilities: no bisected dominos; bisected dominos in rows 1 and 2; bisected dominos in rows 1 and 4; bisected dominos in rows 3 and 2; bisected dominos in rows 3 and 4; and bisected dominos in all four rows.