Math 192r, Problem Set #3: Solutions

1. *Let F_n be the nth Fibonacci number, as Wilf indexes them (with $F_0 = F_1 = 1$, $F_2 = 2$, etc.). Give a simple homogeneous linear recurrence relation satisfied by the sequence whose nth term is...*

   (a) nF_n*:*

   This sequence is given by a formula of the form $Anr^n + Bns^n$ (since $F_n = Ar^n + Bs^n$), where r and s are the roots of $t^2 - t - 1 = 0$. So we need a polynomial which has r as a double root and s as a double root. $(t^2 - t - 1)^2 = t^4 - 2t^3 - t^2 + 2t + 1$ will certainly do. So, writing the nth term of the given sequence as f_n, we have $f_{n+4} = 2f_{n+3} + f_{n+2} - 2f_{n+1} - f_n$.

   Alternatively, we can use generating functions: If $F_0 + F_1x + F_2x^2 + F_3x^3 + \ldots = 1/(1 - x - x^2)$, then, differentiating, we have $1F_1+2F_2x+3F_3x^2+\ldots = (1+2x)/(1-x-x^2)^2$, and the occurrence of $(1 - x - x^2)^2 = 1 - 2x - x^2 + 2x^3 + x^4$ in the denominator tells us that the sequence must satisfy the recurrence $f_{n+4} = 2f_{n+3} + f_{n+2} - 2f_{n+1} - f_n$.

   (b) $1F_1 + 2F_2 + ... + nF_n$*:*

   If we apply the operator $T-I$ to this sequence, we get the sequence considered in part (a). So the sequence f_n whose nth term is $1F_1 + ... + nF_n$ is annihilated by the operator $(T - I)(T^4 - 2T^3 - T^2 + 2T + I) = T^5 - 3T^4 + T^3 + 3T^2 - T - I$.

   Alternatively, we can use generating functions, and multiply the formal power series $(1 + 2x)/(1 - x - x^2)^2$ (considered in the previous sub-problem) by $1 + x + x^2 + \ldots = 1/(1 - x)$. The coefficients of the resulting formal power series are easily seen to be partial sums of exactly the desired kind. So the new denominator is $(1 - x)(1 - x - x^2)^2 = 1 - 3x + x^2 + 3x^3 - x^4 - x^5$, which tells us that the sequence must satisfy the recurrence $f_{n+5} = 3f_{n+4} - f_{n+3} - 3f_{n+2} + f_{n+1} + f_n$.

   (c) $nF_1 + (n-1)F_2 + ... + 2F_{n-1} + F_n$*:* This sum is the coefficient of x^n in the product of the formal power series $F_1x+F_2x^2+...+F_nx^n+...$ with the formal power series $1 + 2x + 3x^2 + ... + nx^{n-1} + ...$ The former is given by a formal power series with denominator $1-x-x^2$ and the latter is given by a formal power series with denominator $(1-x)^2$; when we multiply them, we get a formal power series with denominator $(1-x-x^2)(1-x)^2 = 1-3x+2x^2+x^3-x^4$, so the sequence satisfies the recurrence $f_{n+4} = 3f_{n+3} - 2f_{n+2} - f_{n+1} + f_n$.

(d) *F_n when n is odd, and 2^n when n is even:* We saw in class that the Fibonacci numbers satisfy the recurrence $f_{n+4} = 3f_{n+2} - f_n$. On the other hand, the powers of two satisfy the recurrence $f_{n+2} = 4f_n$. Since any multiple of $T^4 - 3T^2 + I$ annihilates the former, and any multiple of $T^2 - 4I$ annihilates the latter, an operator that annihilates both sequences (while only looking two, four, or six terms earlier) is $(T^4 - 3T^2 + I)(T^2 - 4I) = T^6 - 7T^4 + 13T^2 - 4I$. So $f_{n+6} = 7f_{n+4} - 13f_{n+2} + 4f_n$.

2. *The sequence of polynomials $f_n(x)$ in problem 2 of problem set 1 satisfies a second-order linear recurrence relation with coefficients that are Laurent polynomials in x.*

   (a) *Find it, and prove that it is correct.*
   We will prove that

$$f_{n+1} = (2 + 1/x^2)f_n - f_{n-1} \tag{1}$$

for all $n \geq 2$. Recall that the defining recurrence was

$$f_{n+1} = (f_n^2 + 1)/f_{n-1}. \tag{2}$$

Rather than prove that the sequence of polynomials defined by equation (2) (with the initial conditions $f_0 = f_1 = x$) satisfies (1), we will prove that the sequence of polynomials defined by equation (1) (with the initial conditions $f_0 = f_1 = x$) satisfies (2). For the rest of this proof, $f_0, f_1, \ldots$ denotes the sequence given by recurrence (1).

To show that (2) holds, we must prove that $f_{n+1}f_{n-1} = f_n^2 + 1$. Replacing f_{n+1} by $(2 + 1/x^2)f_n - f_{n-1}$ in this equation, we can rewrite the desired equality in the form

$$f_n^2 + f_{n-1}^2 + 1 = (2 + 1/x^2)f_n f_{n-1}. \tag{3}$$

We will prove this by induction. If $n = 1$, it is simple to check the truth of (3) directly. Otherwise, we may assume as an induction hypothesis that

$$f_{n-1}^2 + f_{n-2}^2 + 1 = (2 + 1/x^2) f_{n-1} f_{n-2}. \qquad (4)$$

To derive (3) from (4), substitute $f_n = (2 + 1/x^2) f_{n-1} - f_{n-2}$ into (3) to obtain

$$\begin{gathered}((2 + 1/x^2) f_{n-1} - f_{n-2})^2 + f_{n-1}^2 + 1 = \\ (2 + 1/x^2)((2 + 1/x^2) f_{n-1} - f_{n-2}) f_{n-1};\end{gathered}$$

expanding and cancelling, we get

$$-2(2 + 1/x^2) f_{n-1} f_{n-2} + f_{n-2}^2 + f_{n-1}^2 + 1 = -(2 + 1/x^2) f_{n-1} f_{n-2}$$

or

$$f_{n-2}^2 + f_{n-1}^2 + 1 = (2 + 1/x^2) f_{n-1} f_{n-2},$$

which is (4). That is, (3) is algebraically equivalent to (4), subject to the substitution $f_n = (2 + 1/x^2) f_{n-1} - f_{n-2}$. Hence (4) implies (3), and the claim follows by induction.

(It may also be possible to prove that the sequence defined by (2) satisfies (1), but I don't see a way to do it.)

(b) *Express $\sum_{n=0}^{\infty} f_n(x) y^n$ as a rational function of x and y.*

Call this generating function $F(x, y)$. Multiplying $F(x, y) = x + xy + \ldots$ by $1 - (2 + 1/x^2)y + y^2$ and using the recurrence relation proved above, we have $(1-(2+1/x^2)y+y^2)F(x, y) = x-(x+1/x)y$, so that

$$F(x, y) = \frac{x - (x + 1/x)y}{1 - (2 + 1/x^2)y + y^2}.$$

We can check this: If we tell Maple

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expand(taylor((x-(x+1/x)*y)/(1-(2+1/x^2)*y+y^2),y,5));
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we get the expected answer.

(Technical aside: The above calculation is rigorously understood to be taking place in the ring of formal power series in the variable y in which the coefficient ring is the ring of all rational functions in the variable x. It can be shown that in this ring, any element whose constant term is 1 (a priori the constant term could be any rational function of x) has a multiplicative inverse, so the quotient makes sense. Indeed, it would also be acceptable to write the generating function as

$$\frac{x^2-(x^2+1)y}{x^2-(2x^2+1)y+x^2y^2}$$

because the denominator of this expression, too, has a multiplicative inverse in the ring of formal power series being considered.)

Incidentally, with recurrence (1) in hand it is easy to prove that

$$f_n=\sum_{k=1}^{n}\binom{n-2+k}{2k-2}x^{3-2k}.$$

Indeed, assuming (for purposes of induction) that this formula holds for f_{n-1} and f_{n-2}, we have

$$\begin{aligned}
f_n &= (2+1/x^2)f_{n-1}-f_{n-2}\\
&= 2\sum_{k=1}^{n-1}\binom{n-3+k}{2k-2}x^{3-2k}+\sum_{k=1}^{n-1}\binom{n-3+k}{2k-2}x^{1-2k}\\
&\quad -\sum_{k=1}^{n-2}\binom{n-4+k}{2k-2}x^{3-2k}\\
&= \sum_{k=1}^{n-1}2\binom{n-3+k}{2k-2}x^{3-2k}+\sum_{k=2}^{n}\binom{n-4+k}{2k-4}x^{3-2k}\\
&\quad -\sum_{k=1}^{n-2}\binom{n-4+k}{2k-2}x^{3-2k}\\
&= \sum_{k=1}^{n}\binom{n-2+k}{2k-2}x^{3-2k}.
\end{aligned}$$

The last equality requires some checking, coefficient by coefficient, and the analysis splits into several cases. For $k=1$, we have

$$2\binom{n-2}{0}-\binom{n-3}{0}=\binom{n-1}{0}$$

which is just $2 - 1 = 1$; for $k = n - 1$, we have

$$2\binom{2n-4}{2n-4} + \binom{2n-5}{2n-6} = \binom{2n-3}{2n-4}$$

which is just $2 + (2n-5) = (2n-3)$; for $k = n$, we have

$$\binom{2n-4}{2n-4} = \binom{2n-2}{2n-2}$$

which is just $1 = 1$; and for $1 < k < n - 1$, we have

$$2\binom{n-3+k}{2k-2} + \binom{n-4+k}{2k-4} - \binom{n-4+k}{2k-2} = \binom{n-2+k}{2k-2},$$

which can be proved by successively substituting

$$\binom{n-2+k}{2k-2} = \binom{n-3+k}{2k-2} + \binom{n-3+k}{2k-3},$$

$$\binom{n-3+k}{2k-2} = \binom{n-4+k}{2k-2} + \binom{n-4+k}{2k-3},$$

and

$$\binom{n-3+k}{2k-3} = \binom{n-4+k}{2k-3} + \binom{n-4+k}{2k-4}.$$