Math 192r, Problem Set #2: Solutions

1. *Find necessary and sufficient conditions for the infinite product $f_1 f_2 f_3 \cdots$ to converge in the ring of formal power series. Be sure to take proper care of degenerate cases, e.g., where one of the f_n's actually is 0. A detailed proof is not required.*

   There are three ways for such a product to converge. First, one of the f_n's could be 0, in which case the product is 0. Second, infinitely many of the f_n's could be of positive order (i.e., could have constant term 0), in which case the product is again 0. Third, it could be the case that $f_n \to 1$, i.e., that $\operatorname{ord}(f_n - 1) \to \infty$, in which case the infinite product still converges (to something non-zero, unless one of the factors is zero as in the first case).

   Here is a more rigorous analysis. If f_n is 0 for some n, the infinite product converges to 0. Now suppose none of the f_n's is 0; then each factor in the infinite product has finite order. Suppose that infinitely many of the factors have positive order (i.e., are divisible by x). Then the infinite product again converges to 0. On the other hand, suppose (still assuming that all of the f_n's are non-zero) that only finitely many of the factors have positive order; that is, suppose that all but finitely many have order 0. If we let N denote the sum of the orders of those factors, then we find that $f_1 f_2 \cdots f_n$ is never divisible by x^{N+1}, no matter how big n is, which means that the infinite product cannot converge to 0.

   Now, suppose $f_1 f_2 \cdots f_n \to g$ for some non-zero formal power series g. For convenience, we will write the partial product $f_1 \cdots f_n$ as F_n. Since $F_{n-1} \to g$, we may appeal to the continuity of division to infer that $f_n = F_n / F_{n-1} \to 1$. Conversely, suppose $f_n \to 1$, so that $\operatorname{ord}(f_n - 1) \to \infty$. The relation $F_n - F_{n-1} = F_{n-1}(f_n - 1)$ gives $\operatorname{ord}(F_n - F_{n-1}) \geq \operatorname{ord}(f_n - 1)$, and since $\operatorname{ord}(f_n - 1) \to \infty$, we have $\operatorname{ord}(F_n - F_{n-1}) \to \infty$ as well (or equivalently $F_n - F_{n-1} \to 0$). But this implies that the infinite series $F_1 + (F_2 - F_1) + (F_3 - F_2) + \cdots$ is convergent, which (by telescoping) is tantamount to the assertion that the sequence $F_1, F_2, F_3, \ldots$ converges, which means that the infinite product $f_1 f_2 f_3 \cdots$ converges.

2. *By comparing the expansions of* $1/(1-x-x^2)$ *and* $1/(1-x-y)$*, derive a formula for the Fibonacci numbers as sums of binomial coefficients.*

Specializing $1/(1-x-y) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$ to $y = x^2$, we get $1/(1-x-x^2) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} x^{2n-k}$. So the Fibonacci number F_m, being equal to the coefficient of x^m in the formal power series expansion of $1/(1-x-x^2)$, must also be equal to $\sum \binom{n}{k}$, where the sum is taken over all n, k with $0 \leq k \leq n$ and $2n - k = m$. These pairs are $(n,k) = (m,m), (m-1, m-2), (m-2, m-4), \ldots$, so $F_m = \binom{m}{m} + \binom{m-1}{m-2} + \binom{m-2}{m-4} + \ldots$, where the sum stops whenever the lower index is 0 (for m even) or 1 (for m odd). Example: $F_4 = \binom{4}{4} + \binom{3}{2} + \binom{2}{0} = 1 + 3 + 1 = 5$ and $F_5 = \binom{5}{5} + \binom{4}{3} + \binom{3}{1} = 1 + 4 + 3 = 8$.