Math 192r, Problem Set #19: Solutions

1. *In problem #3 of assignment #17, multivariate polynomials*

$$D(x_1, x_3, \ldots, x_{2n+1}; y_2, y_4, \ldots, y_{2n})$$

were defined. Find an infinite acyclic directed graph with special vertices $\ldots, v_{-1}, v_0, v_1, \ldots$ *where all edges are assigned weight 1 and vertices are assigned weights according to some scheme that you must devise, so that for all integers* $i \leq j$ *the sum of the weights of the paths from* v_i *to* v_j *is* $D(x_{2i+1}, x_{2i+3}, \ldots, x_{2j+1}; y_{2i+2}, y_{2i+4}, \ldots, y_{2j})$. *Include a proof that your answer is correct.*

We can build a model of our directed graph in $\mathbf{Z}^2$. Our vertices will be pairs (i, j) with $i - j$ equal to 0, 1, 2, or 3, and there will be a directed edge from vertex (i, j) to vertex (i', j') if and only if $(i' - i, j' - j) = (1, 0)$ or $(0, 1)$. The special vertices are $v_i = (i, i)$. The weight of vertex (i, j) is x_{i+j} if $i - j = 0$, $y_{i+j}/x_{i+j-1}x_{i+j+1}$ if $i - j = 1$, $x_{i+j}/y_{i+j-1}y_{i+j+1}$ if $i - j = 2$, and y_{i+j} if $i - j = 3$. Let $W_{i,j}$ be the sum of the weights of the paths from v_i to v_j. It is easy to check that $W_{i,i} = x_{2i}$ and $W_{i,i+1} = y_{2i+1}$ as required. To prove that $W_{i,j} = D(x_{2i+1}, \ldots, x_{2j+1}; y_{2i+2}, \ldots, y_{2j})$ for all values of i and j, it suffices to verify that the numbers $W_{i,j}$ satisfy the diamond recurrence. That is, we must show that $W_{i,j}W_{i+1,j-1} - W_{i+1,j}W_{i,j-1} = 1$. By Lindstrom's lemma, this is equivalent to the assertion that the signed sum of the weights of the 2-routings from $\{v_i, v_{i+1}\}$ to $\{v_j, v_{j-1}\}$ is 1. But we already know that there is a unique 2-routing of this kind, connecting v_i to v_j and v_{i+1} to v_{j-1}, so it suffices to check that the weight of this 2-routing is 1. To check this, note that the product of the weights of (i, i), $(i + 1, i)$, $(i + 1, i + 1)$, $(i + 2, i)$, $(i + 2, i + 1)$, and $(i + 2, i + 2)$ is $(x_{2i})(y_{2i+1}/x_{2i}x_{2i+2})(x_{2i+2})(x_{2i+2}/y_{2i+1}y_{2i+3})(y_{2i+3}/x_{2i+2}x_{2i+4})(x_{2i+4}) = 1$, the product of the weights of $(i+3, i)$, $(i+3, i+1)$, $(i+3, i+2)$, and $(i+3, i+3)$ is $(y_{2i+3})(x_{2i+4}/y_{2i+3}y_{2i+5})(y_{2i+5}/x_{2i+4}x_{2i+6})(x_{2i+6}) = 1$, the product of the weights of $(i + 4, i + 1)$, $(i + 4, i + 2)$, $(i + 4, i + 3)$, and $(i + 4, i + 4)$ is 1, etc.

2. *Consider an infinite array with tilted upper boundary like the one shown*

below:

$$
\begin{array}{cccccccccccccccc}
 & & & & & & & & & & & & & & & \vdots \\
 & & & & & & & & & & & & & & x_5 & \\
 & & & & & & & & & & & x_4 & & w_5 & & y_5 \\
 & & & & & & & & x_3 & & w_4 & & y_4 & & * & \\
 & & & & & x_2 & & w_3 & & y_3 & & * & & * & & * \\
 & & x_1 & & w_2 & & y_2 & & * & & * & & * & & * & \\
 & w_1 & & y_1 & & * & & * & & * & & * & & * & & * \\
\vdots & & & & & & & & & & & & & & & \vdots
\end{array}
$$

Here the entries w_i, x_i, y_i are formal indeterminates, and the entries marked with asterisks are determined by the diamond rule as in assignment #17; that is, whenever the array contains four entries arranged like

$$
\begin{array}{ccc}
 & a & \\
b & & c \\
 & d &
\end{array}
$$

we must have $ad - bc = 1$. Some experimentation will probably convince you that each entry in the table is a Laurent polynomial in the variables w_i, x_i, y_i, and that moreover each coefficient in this polynomial equals $+1$. Show how for each such Laurent polynomial, the Laurent monomials that participate correspond to the perfect matchings of some graph (just as was the case in assignment #17). Give a concrete description of the graphs and the correspondence between matchings and monomials (including either a proof or convincingly large examples).

Consider first what we get when we specialize all the variables to equal 1:

$$
\begin{array}{cccccccccccccccc}
 & & & & & & & & & & & & & & & \vdots \\
 & & & & & & & & & & & & & & 1 & \\
 & & & & & & & & & & & \mathbf{1} & & 1 & & 1 \\
 & & & & & & & & 1 & & 1 & & 1 & & 2 & \\
 & & & & & \mathbf{1} & & 1 & & 1 & & 2 & & 3 & & 7 \\
 & & 1 & & 1 & & 1 & & 2 & & 3 & & 7 & & 11 & \\
 & 1 & & 1 & & 2 & & 3 & & 7 & & 11 & & 26 & & 41 \\
\vdots & & & & & & & & & & & & & & & \vdots
\end{array}
$$

I claim that each of these entries counts the number of lattice paths joining two points on the upper boundary of 1's, where a lattice path may pass only through vertices that are marked with 1's, 2's, 3's, and 7's (call these the "single-digit locations"). Specifically, given a location in the table, trace a diagonal going northwest until you hit the boundary, and call that location i; likewise, trace a diagonal going northeast until you hit the boundary, and call that location j. Then I claim that the entry in question is equal to the number (call it $N(i,j)$) of paths from location i to location j consisting of northeast and southeast steps, by way of single-digit locations.

For example, I claim that the 3 in the last row of the above excerpt counts the lattice paths between the two boldface 1's on the boundary. To see why this is true, take four entries that form a diamond, and let the associated locations on the boundary be i, i', j', j (from left to right). To get a proof by induction that the entries count lattice paths, we need to verify that $N(i,j)N(i',j') - N(i,j')N(i',j) = 1$. But (by Lindstrom's lemma) this is a consequent of the fact that there is a unique 2-routing that joins $\{i,i'\}$ with $\{j,j'\}$, and that it connects i with j and i' with j'. (That's because the single-digit locations in any column are precisely the top two locations in that column.)

We have now shown that the entries in the table count lattice paths. But a lattice path is just a special case of a routing (namely, a 1-routing), so by using the routings-into-matchings trick we can get a bijection between lattice paths in that graph and perfect matchings of a certain graph. For instance, consider the directed graph

```
                  o
                 /
            o   o
           / \ /
      o   o   o
     / \ / \ /
o   o   o   o
 \ / \ / \ /
  o   o   o
   \ / \ /
    o   o
```

The 11 lattice paths joining the leftmost and rightmost vertices in this directed graph correspond to the 11 perfect matchings of the graph

```
                                  o
                                 /
                      o-o     o-o
                     /   \   /
          o-o     o-o     o-o
         /   \   /   \   /
o     o-o     o-o     o-o
 \   /   \   /   \   /
  o-o     o-o     o-o
     \   /   \   /
      o-o     o-o
```

which in turn correspond to the 11 perfect matchings of the graph

```
                      o-o
                     /   \
          o-o     o-o     o
         /   \   /   \   /
      o-o     o-o     o-o
     /   \   /   \   /
    o     o-o     o-o
     \   /   \   /
      o-o     o-o
```

Now we may bring the variables w_i, x_i, y_i back into the story. If indeed the rational functions that are generated by the two-dimensional recurrence are Laurent polynomials, and if indeed each monomial contributing to these polynomials has coefficient $+1$, then we can say that the number of monomials must go like 1, 1, 1, 2, 3, 7, 11, 26, 41, $\ldots$, and they should correspond to matchings of the above graph.

To see how the correspondence goes, label the hexagonal cells as shown in the diagram on the next page:

```
                    o-o
                   /   \
        o-o     o-o x_3 o
       /   \   /   \   /
    o-o x_2 o-o w_3 o-o y_3
   /   \   /   \   /
  o w_2 o-o y_2 o-o
   \   /   \   /
y_1 o-o     o-o
```

(Note that we've added two extra "ghost-cells" at the ends.) Given a matching μ of the graph (and its associated monomial) and a cell c in the graph (and its associated variable), the exponent of the variable in the monomial is equal to 2 minus the number of edges in μ adjacent to c, unless c is a ghost-cell, in which case the exponent of the variable in the monomial is equal to 1 minus the number of edges in μ adjacent to c. For instance, the matching

```
                    o-o

        o-o     o-o x_3 o
                       /
    o o x_2 o o w_3 o o y_3
   /   \   /   \   /
  o w_2 o o y_2 o o

y_1 o-o     o-o
```

corresponds to the monomial

$$y_1^{1-0} w_2^{2-3} x_2^{2-3} y_2^{2-3} w_3^{2-3} x_3^{2-2} y_3^{1-1} = \frac{y_1}{w_2 x_2 y_2 w_3}.$$