Math 192r, Problem Set #16: Solutions

1. *Use Dodgson condensation to prove the Vandermonde determinant formula*
$$\det(M) = \prod_{1 \le i < j \le n} (x_j - x_i)$$
where M is the n-by-n matrix whose i, jth entry (for $1 \le i, j \le n$) is x_j^{i-1}.

   The claim is easily verified for $n = 0$ and $n = 1$. We will prove higher cases by induction; hence hereafter we will assume that the formula is true for $n-1$ and $n-2$ in order to prove it for $n > 1$. Write the determinant of M as $d_n(x_1, x_2, \ldots, x_n)$. If we apply Dodgson condensation to this matrix, we get
$$\begin{aligned} d_n(x_1, \ldots, x_n) e_{n-2}(x_2, \ldots, x_{n-1}) &= d_{n-1}(x_1, \ldots, x_{n-1}) e_{n-1}(x_2, \ldots, x_n) \\ &\quad - d_{n-1}(x_2, \ldots, x_n) e_{n-1}(x_1, \ldots, x_{n-1}), \end{aligned}$$
where $e_m(y_1, \ldots, y_m)$ is the determinant of the m-by-m matrix whose i, jth entry is x_j^i. By the multilinearity of the determinant,
$$e_m(y_1, \ldots, y_m) = y_1 \cdots y_m \, d_m(y_1, \ldots, y_m),$$
so the condensation relation may be written as
$$x_2 \cdots x_{n-1} D_1^n D_2^{n-1} = x_2 \cdots x_n D_1^{n-1} D_2^n - x_1 \cdots x_{n-1} D_2^n D_1^{n-1},$$
where D_i^j is short for $d_{j-i+1}(x_i, \ldots, x_j)$. Simplifying, we obtain
$$D_1^n D_2^{n-1} = x_n D_1^{n-1} D_2^n - x_1 D_2^n D_1^{n-1}$$
or
$$D_1^n D_2^{n-1} = (x_n - x_1) D_1^{n-1} D_2^n.$$
Our induction hypothesis implies that $D_1^{n-1} = \left(\prod_{j=2}^{n-1} (v_j - v_1)\right) D_2^{n-1}$ and $D_2^n = \left(\prod_{i=2}^{n-1} (v_n - v_i)\right) D_2^{n-1}$, so the right hand side of the preceding inset equation can be rewritten as
$$(x_n - x_1) \left(\prod_{j=2}^{n-1} (v_j - v_1) \right) \left(D_2^{n-1} \right) \left(\prod_{i=2}^{n-1} (v_n - v_i) \right) \left(D_2^{n-1} \right).$$

Hence

$$\begin{aligned}
D_1^n &= (x_n - x_1)\left(\prod_{j=2}^{n-1}(v_j - v_1)\right)\left(\prod_{i=2}^{n-1}(v_n - v_i)\right) D_2^{n-1} \\
&= (x_n - x_1)\left(\prod_{j=2}^{n-1}(v_j - v_1)\right)\left(\prod_{i=2}^{n-1}(v_n - v_i)\right)\left(\prod_{2\le i<j\le n-1}(v_j - v_i)\right) \\
&= \prod_{1\le i<j\le n}(v_j - v_i),
\end{aligned}$$

which was to be proved.

Someone reading the above might object that there is a flaw in the proof, inasmuch as we are dividing by an expression D_2^n that can vanish (and indeed will vanish if any two of $x_2, \ldots, x_{n-1}$ are equal), and that it is invalid to divide by zero. But this objection has no force, since we our conducting our proof in the realm of formal polynomials and formal rational functions. In the field of rational functions in the two variables x and y, the expressions $(x^2 - y^2)/(x - y)$ and $x + y$ are actually equal; one does not substitute actual values for x and y, so one need not worry about "what if" $x = y$.

On the other hand, once one has proved that a pair of polynomials are equal as formal expressions (namely, the determinant of a certain matrix and a certain product of differences), one can treat the two polynomials as functions and make substitutions for the variables. So the Vandermonde identity that we have proved isn't just true for one particular n-by-n matrix in the ring of polynomials in the variables $x_1, \ldots, x_n$; it's true for every n-by-n matrix whose entries are particular values satisfying certain relations.

2. *Using Dodgson condensation, Lindstrom's lemma, and the bijection between tilings and routings discussed in class, prove that for all $a, b, c \geq 0$, the number of ways to tile an a, b, c, a, b, c semiregular hexagon with unit rhombuses is equal to*

$$\frac{H(a+b+c)H(a)H(b)H(c)}{H(a+b)H(a+c)H(b+c)}$$

where $H(0) = H(1) = 1$ and $H(n) = 1!2!3!\cdots(n-1)!$ for $n > 1$.

Let $T(a,b,c)$ denote the number of rhombus tilings of the a,b,c,a,b,c semiregular hexagon. It is easy to check that for all $a,b \geq 0$, $T(a,b,0) = 1 = \frac{H(a+b+0)H(a)H(b)H(0)}{H(a+b)H(a+0)H(b+0)}$ and $T(a,b,1) = \frac{(a+b)!}{(a)!(b)!} = \frac{H(a+b+1)/H(a+b)}{(H(a+1)/H(a))(H(b+1)/H(b)}$ $= \frac{H(a+b+1)H(a)H(b)H(1)}{H(a+b)H(a+1)H(b+1)}$. We will prove the claim for $c > 1$ using induction on c.

Rhombus-tilings of the a,b,c,a,b,c semiregular hexagon correspond to routings with c sources and c sinks in a directed graph in which the number of paths from the ith source to the jth sink equals $\binom{a+b}{b-i+j}$. Therefore by Lindstrom's lemma we have $T(a,b,c) = \det M(a,b,c)$ where $M(a,b,c)$ denotes the c-by-c matrix whose i,jth entry is $\binom{a+b}{b-i+j}$. In view of the this, Dodgson condensation tells us that

$$\begin{aligned} T(a,b,c)T(a,b,c-2) &= T(a,b,c-1)^2 \\ &\quad -T(a+1,b-1,c-1)T(a-1,b+1,c-1). \end{aligned}$$

For slight notational convenience, I'll re-index this as

$$T(a,b,c+1)T(a,b,c-1) = T(a,b,c)^2 - T(a+1,b-1,c)T(a-1,b+1,c).$$

The problem now reduces to algebraically verifying that $T(a,b,c+1)$ must be given by the $H(\ )$-formula if $T(a,b,c-1)$, $T(a,b,c)$, $T(a+1,b-1,c)$ and $T(a-1,b+1,c)$ are. Equivalently, we must verify that if all five of these $T(\ )$-values are as given by the $H(\ )$-formula, then the expression

$$T(a,b,c+1)T(a,b,c-1) - T(a,b,c)^2 + T(a+1,b-1,c)T(a-1,b+1,c)$$

must vanish.

If we trust Maple, then we can prove this by noting that the final command in the string of commands

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H := proc(n) product(k!,k=1..n-1); end;
T := proc(a,b,c) H(a+b+c)*H(a)*H(b)*H(c)
        /H(a+b)/H(a+c)/H(b+c); end;
U := T(a,b,c)*T(a-2,b,c)-T(a-1,b,c)^2
        +T(a-1,b-1,c+1)*T(a-1,b+1,c-1);
simplify(expand(U));
```

gives the output `0`. However, if you're more skeptical, here's a sketch of how you can show by hand that the expression $T(a,b,c+1)T(a,b,c-1) - T(a,b,c)^2 + T(a+1,b-1,c)T(a-1,b+1,c)$ vanishes when each $T(\ )$ is expanded using the $H(\ )$-formula. Write each of the three terms as a fraction, and in each of the terms divide the numerator by $H(a+b+c-1)H(a+b+c)H(a-1)H(a)H(b-1)H(b)H(c-1)H(c)$ and the denominator by $H(a+b)^2H(a+c-1)H(a+c)H(b+c-1)H(b+c)$, obtaining another messy expression. But we have made progress: where before we had a sum each term of which was a ratio of products each factor of which was a value of the H-function, we now have a sum each term of which is a ratio of products each factor of which is a value of the factorial function, Moreover, there are now some factors common to all three terms; removing them gives

$$\begin{aligned} &\tfrac{(a+b+c)!(a-1)!(b-1)!(c)!}{(a+c)!(b+c)!} \\ -&\tfrac{(a+b+c-1)!(a-1)!(b-1)!(c-1)!}{(a+c-1)!(b+c-1)!} \\ +&\tfrac{(a+b+c-1)!(a)!(b)!(c-1)!}{(a+c)!(b+c)!}. \end{aligned}$$

Removing common factors again gives us

$$(a+b+c-1)(c-1) - (a+c-1)(b+c-1) + (a)(b),$$

which vanishes.