

Math 191 Notes, 2003 October 30

Wallis's product

$$\begin{aligned}\frac{\pi}{2} &\approx \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdots (2m-2)(2m-2)(2m)}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdots (2m-3)(2m-1)(2m-1)} \\ \sqrt{\frac{\pi}{2}} &\approx \frac{2 \cdot 4 \cdot 6 \cdot 8 \cdots (2m-2)\sqrt{2m}}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2m-1)} \\ &= \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdots (2m-2) \cdot 2m \cdot \sqrt{2m}}{1 \cdot 2 \cdot 3 \cdot (2m-1) \cdot 2m \cdot 2m} \\ \sqrt{\frac{\pi}{2}} &\approx \frac{2^{2m}(m!)^2\sqrt{2m}}{(2m)!2m} \\ \frac{(2m)!}{2^{2m}(m!)^2} &\approx \frac{1}{\sqrt{\pi m}}\end{aligned}$$

Then we can use this to evaluate u_{2m} .

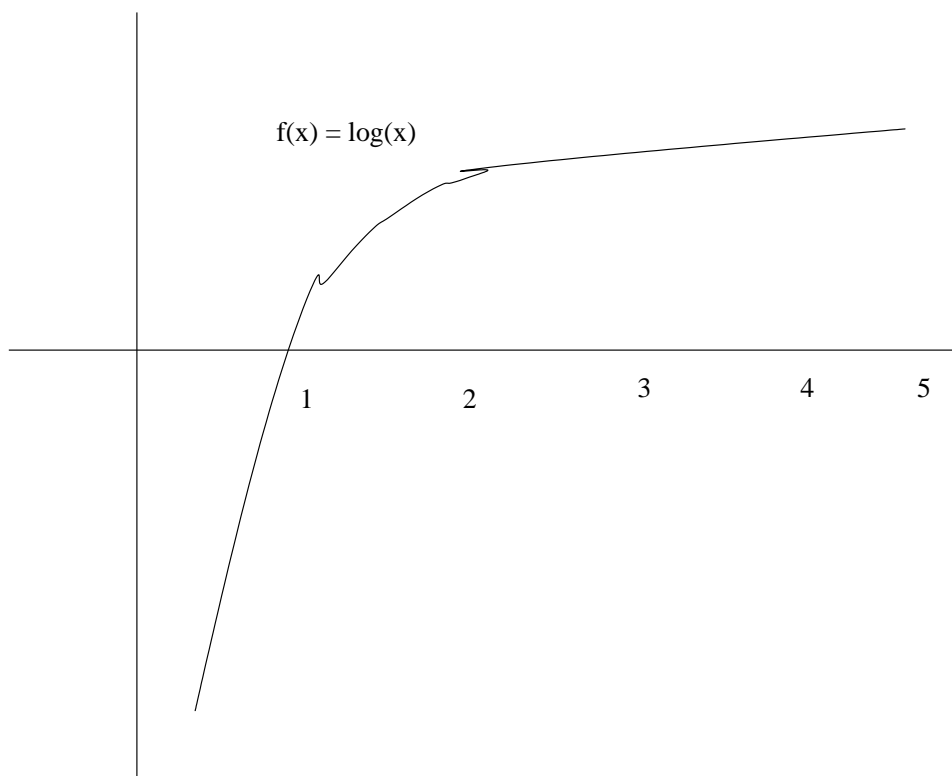
Stirling's Approximation

This is often proved using asymptotic series (very dangerous if we don't know what we're doing). We'll try something half-way there.

$$\log n! = \log 1 + \log 2 + \cdots + \log n$$

There is an approximation to this using an integral. using the points, $\frac{1}{2}, \frac{3}{2}, \dots, n + \frac{1}{2}$.

$$\begin{aligned}\int_{1/2}^{n+1/2} \log x \, dx &= \cdots \\ &= (n + 1/2) \left(\log n + \frac{1}{2n} \right) - n - \frac{1}{2} \log \frac{1}{2} \\ &= (n + 1/2) \log n + 1/2 + 1/4n - n - 1/2 \log 1/2 \\ &\quad + \text{unknown Riemann sum error}\end{aligned}$$



Let's try $\log n! = (n + 1/2) \log n - n + \log K$

$$\begin{aligned}
 n! &= K n^n e^{-n} \sqrt{n} \\
 \frac{(2m)!}{2^{2m} (m!)^2} &\approx \frac{K (2m)^{2m} e^{-2m} \sqrt{2m}}{2^{2m} K^2 m^{2m} e^{-2m} m} \approx \frac{1}{\sqrt{\pi m}} \\
 \frac{Km}{\sqrt{2m}} &= \sqrt{\pi m} \\
 K &= \sqrt{2\pi} \\
 n! &\approx n^n e^{-n} \sqrt{2\pi n}
 \end{aligned}$$

Ok, stirling's approximation is nice. But what about the arcsin? Say we have a sum with a large number of steps, with $0 < x < 1$:

$$\begin{aligned}
 \sum_{k \leq xn} u_{2k} u_{2n-2k} &\approx \sum_{k \leq xn} \frac{1}{\pi \sqrt{k(n-k)}} \\
 &\approx \int_{u=0}^{xn} \frac{du}{\pi \sqrt{u(n-u)}}
 \end{aligned}$$

Set $u = n \sin^2 t$, $du = 2n \sin t \cos t$.

$$\begin{aligned}\sqrt{u(n-u)} &= \sqrt{n \sin^2 t \cdot n \cos^2 t} = n \sin t \cos t \\ xn &= n \sin^2 t \\ t &= \arcsin \sqrt{x} \\ \int_{u=0}^{xn} \frac{du}{\pi \sqrt{u(n-u)}} &= \int_0^{\arcsin \sqrt{x}} \frac{2dt}{\pi} = \frac{2}{\pi} \arcsin \sqrt{x}\end{aligned}$$

What does this look like?

$$\frac{2}{\pi} \arcsin \sqrt{x} = \begin{cases} 0 & x = 0 \\ 1 & x = 1 \\ 1/2 & x = 1/2 \\ 1/3 & x = 1/4 \end{cases}$$

Has the interesting property that probability of being in the middle half is $1/3$. The graph of this is a sine curve on its side.

We are now in continuous probability distributions! We've taken the limiting case of a discrete distribution, and arrived at this.

Continuous distribution functions

$$F(x) = \int_{-\infty}^x f(u) du$$

This is the definition—integral of the density function instead of sum of the mass function. Obvious choice of the density function is $f(x) = F'(x)$. Of course, if we have a finite number of removable discontinuities, the integral $F(x)$ need not change.

What about the relevant σ -fields? It's easy to calculate the following probability:

$$\mathbb{P}(a \leq X \leq b) = \int_a^b f(x) dx$$

Imagine a random variable $Y = g(X)$ given by some periodic graph that oscillates like a sign wave. Then we ask for which x , is Y greater than a certain point. This is a countably infinite union of sets. Let's try the definition:

$$\mathbb{P}(B) = \int_B f(x) dx \text{ where } B \text{ is a union of open intervals}$$

One gotcha: what about the σ -field? The complements of this are closed intervals! Is B in the σ -field? Is there a way to define this? Well, it's named \mathcal{B} , it's called the Borel σ -field, and it's the "smallest" σ -field that contains all open intervals.

Expectations

We have a continuous random variable X whose expectation we want to calculate.

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f(x) dx$$

Silly example: X is uniform on the interval $[1, 3]$.

$$\int_1^3 x \cdot \frac{1}{2} dx = \frac{x^2}{2} \Big|_1^3 = 2$$

Law of Unconscious Statistician

How about $\mathbb{E}(X^3)$? The idea is that if $Y = g(x)$, then

$$\mathbb{E}(Y) = \int_{-\infty}^{\infty} g(x) f(x) dx$$

. Proof: suppose that $Y = g(X)$ where $g(x) \geq 0$ for all x .

Lemma:

$$\begin{aligned} \int_0^{\infty} \mathbb{P}(X > y) dy &= \int_0^{\infty} dy \int_y^{\infty} f(x) dx = \int \int_R f(x) dx dy \\ &= \int_0^{\infty} dx f(x) \int_0^x dy = \int_0^{\infty} x f(x) dx = \mathbb{E}(X) \end{aligned}$$

Proof of unconscious statistician: Let $Y = g(X)$, $g(x) \geq 0$ for all x and $f(x) = 0$ for $x < 0$.

$$\mathbb{E}(Y) = \int_0^{\infty} \mathbb{P}(g(X) > y) dy$$

Find all places where $g(x) > y$, these are the intervals on the graph which are above the horizontal line y , call this $B(y)$. So then we have:

$$\begin{aligned} \mathbb{E}(Y) &= \int_0^{\infty} dy \int_{B(y)} f_X(x) dx = \int \int_R f_X(x) dx dy \\ &= \int_0^{\infty} dx f_X(x) \int_0^{g(x)} dy = \int_0^{\infty} g(x) f_X(x) dx \end{aligned}$$