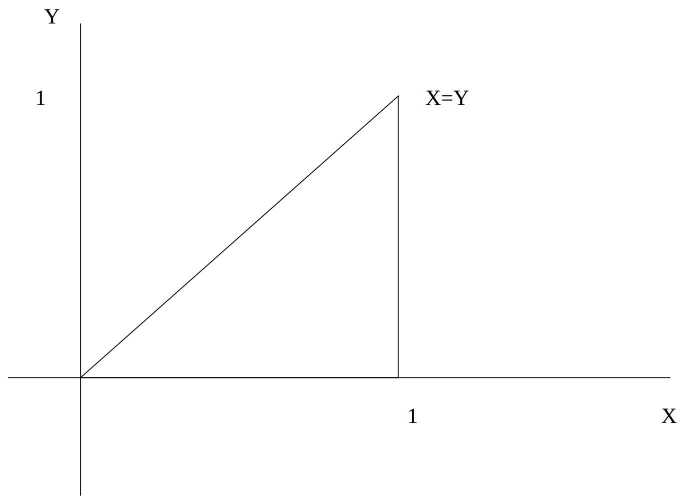
Math 191 Notes, 2003 December 2

Continuation of example last time

X, Y independent, uniform on $[0, l]$. What is $\mathbb{E}(|X - Y|)$?

1. Straightforward method: Let $0 \le Y \le X \le 1$ and compute $\mathbb{E}(X - Y)$. Not trivial, because now X and Y are no longer independent!



$$f_{X,Y}(x, y) = 2$$

$$\begin{aligned}\int_0^1 2dx \int_0^y dy(x-y) &= \int_0^1 2xdx \int_0^x dy - \int_0^1 2dx \int_0^x ydy \\ &= \int_0^1 2x^2 dx - \int_0^1 x^2 dx = \int_0^1 x^2 dx = \frac{1}{3}\end{aligned}$$

2. The answer we want should be linear in l (make sure you understand this!). $\mathbb{E}(|X - Y|) = \alpha l$. Divide the segment into two halves: the probability that they happen on different halves is $\frac{1}{2}$, as is the probability of them being in the same half.

$$\begin{aligned}\alpha l &= \underbrace{\frac{1}{2} \cdot \frac{1}{2} l}_{\text{same half}} + \underbrace{\frac{1}{2} \cdot \frac{1}{2} \alpha l}_{\text{opposite halves}} \\ \frac{3}{4}\alpha l &= \frac{1}{4} l \\ \alpha &= \frac{1}{3}\end{aligned}$$

3. Crofton method: turning a continuous probability problem into a differential equation and solving that. Imagine that the integral runs from 0 to z, and imagine what would be different if it ran from 0 to $z+h$. Here is the clever idea:

$$
\begin{aligned}
\mathbb{P}(X \in [0,z]) &= \frac{z}{z+h} \\
\mathbb{P}(X, Y \in [0,z]) &= \left(\frac{z}{z+h}\right)^2 \\
&= \frac{1}{\left(1+\frac{h}{z}\right)^2} \approx 1 - \frac{2h}{z} + \cdots \\
\mathbb{E}(|X-Y|) &= L(z) \\
L(z+h) &= \left(1 - \frac{2h}{z}\right) L(z) + \frac{2h}{z} \cdot \frac{z+h}{2} \\
L(z+h) - L(z) + \frac{2h}{z} L(z) &= \frac{2h}{z} \cdot \frac{z}{2} \\
\frac{L(z+h) - L(z)}{h} + \frac{2L(z)}{z} &= 1
\end{aligned}
$$

Note that everything here is to first-order in h, so we throw away any h^2 terms. Now we take the limit as $h \to 0$,

$$\frac{dL}{dz} + \frac{2L}{z} = 1$$

Separation of variables is not directly possible, but don't despair! Introduce some integrating factor $\mu(z)$ that will get rid of the annoying second term:

$$
\begin{aligned}
\mu \frac{dL}{dz} + \mu \frac{2L}{z} &= \mu \\
\frac{d}{dz}(\mu L) - L \frac{d\mu}{dz} + \mu \frac{2L}{z} &= \mu
\end{aligned}
$$

Now, we don't want the last two terms on the left-hand side, so set their values equal to each other…

$$\begin{aligned} L\frac{d\mu}{dz} &= \mu\frac{2L}{z} \\ \frac{d\mu}{\mu} &= \frac{2dz}{z} \\ \log\mu &= 2\log z \\ \mu &= z^2 \end{aligned}$$

Now, let's go back to our original equation and plug it in:

$$\begin{aligned} z^2\left(\frac{dL}{dz} + \frac{2L}{z}\right) &= z^2 \\ \frac{d}{dz}\left(Lz^2\right) &= z^2 \\ Lz^2 &= \frac{z^3}{3} + C \end{aligned}$$

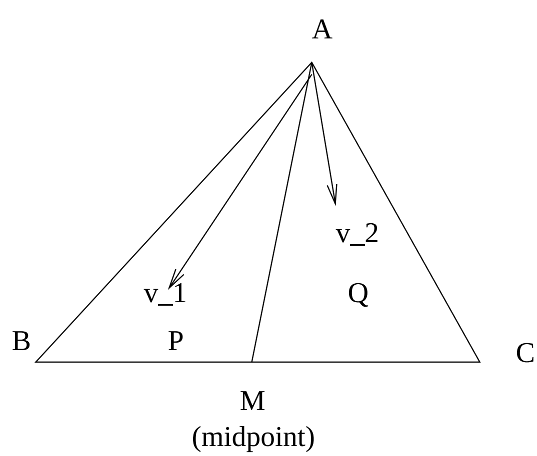
Boundary condition gives us $z = 0 \Rightarrow L(z) = 0$, so $C = 0$, so

$$L = \frac{z}{3}$$

The quiz will have something like this, using Crofton's method.

Triangle problem

Choose randoom point $\vec{v}_1$ going to point P, in $\triangle ABM$, $\vec{v}_2$ going to Q in $\triangle AMC$.

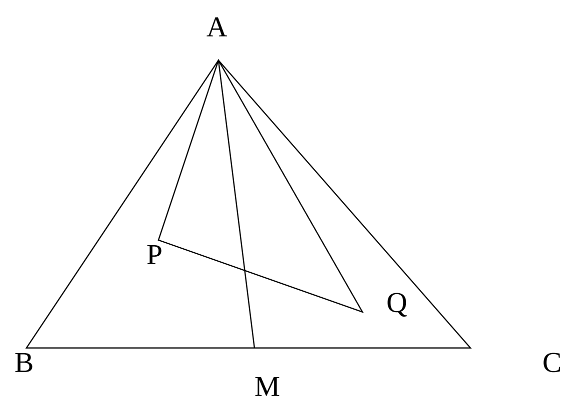


$$
\begin{aligned}
\text{Area of } |APQ| &= \frac{1}{2}|\vec{v}_1 \times \vec{v}_2| \\
&= \frac{1}{2}|x_1y_2 - x_2y_1| = \frac{1}{2}(x_1y_2 - x_2y_1) \\
\mathbb{E}(\text{Area}) &= \frac{1}{2}(\mathbb{E}(x_1)\mathbb{E}(y_2) - \mathbb{E}(x_2)\mathbb{E}(y_1))
\end{aligned}
$$

The expected value for the location of a point inside the triangle is at the center of mass of the triangle. Let the center of mass of $\triangle ABM$ be $\vec{G}_1$, and of $\triangle AMC$ to be $\vec{G}_2$:

$$
\begin{aligned}
&= \frac{1}{2}|\vec{G}_1 \times \vec{G}_2| = \text{Area}(AG_1G_2) \\
&= \frac{1}{2}(\text{base})(\text{altitude}) = \frac{1}{2} \cdot \frac{1}{3}b \cdot \frac{2}{3}h \\
&= \frac{2}{9}\left(\frac{1}{2}bh\right) = \frac{2}{9}|ABC|
\end{aligned}
$$

Now, let's do the original problem of picking two random points anywhere in the triangle.

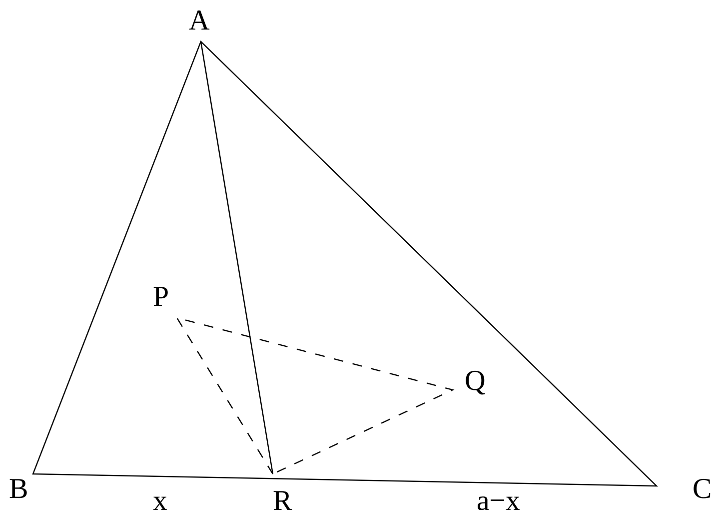


$$\mathbb{E}(|APQ|) = \alpha|ABC|$$

The probability that P, Q are in different halves or in the same half are both $p = \frac{1}{2}$.

$$
\begin{aligned}
\alpha|ABC| &= \frac{1}{2} \cdot \frac{1}{2}\alpha|ABC| + \frac{1}{2} \cdot \frac{2}{9}|ABC| \\
\alpha &= \frac{4}{27}
\end{aligned}
$$

Another method: Crofton's method. Lemma with 3 Cases:



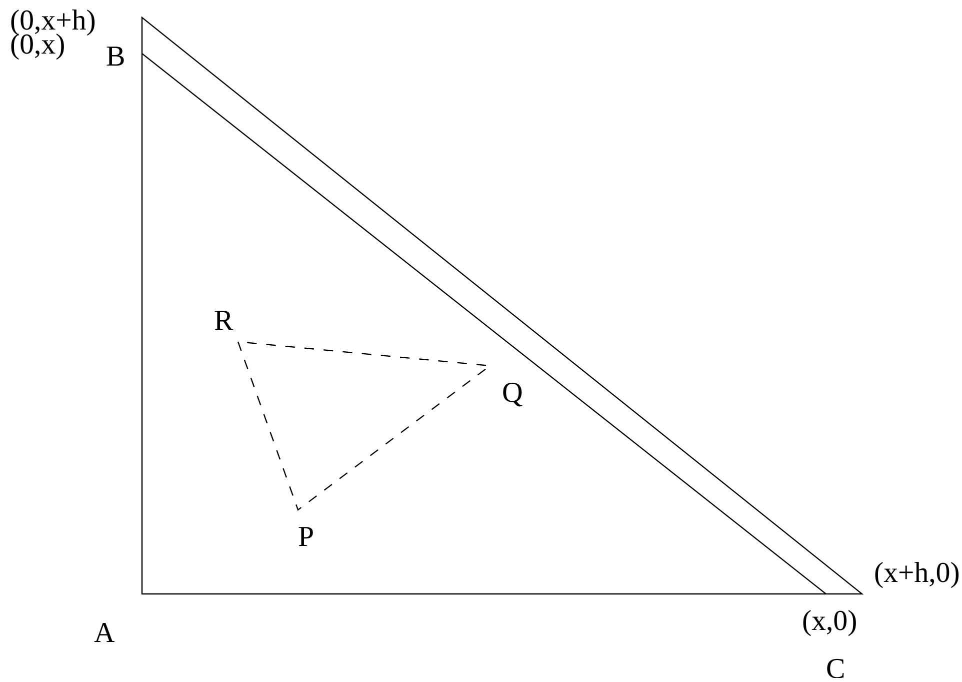
1. P, Q both in left triangle, $p = (x/a)^2$
2. P, Q both in right triangle, $p = \left(\frac{a-x}{x}\right)^2$
3. P, Q in different halves $p = \frac{2x(a-x)}{a^2}$

$$\mathbb{E}(|RPQ|) = \mathbb{E}\left(\underbrace{\frac{x}{a}^2 \cdot \frac{4}{27}\frac{x}{a}|ABC|}_{\#1} + \underbrace{\left(\frac{a-x}{a}\right)^2 \cdot \frac{4}{27}\frac{a-x}{a}|ABC|}_{\#2} + \underbrace{\frac{2x(a-x)}{a^2}\frac{1}{2} \cdot \frac{2}{9}|ABC|}_{\#3}\right)$$

where the extra factor of $1/2$ in the last term is because we are using the bottom as the altitude instead of the top.

$$\mathbb{E}(|RPQ|) = \frac{1}{27}|ABC| + \frac{1}{27}|ABC| + \frac{1}{27}|ABC| = \frac{1}{9}|ABC|$$

where we calculated each of the integrals individually. Now we're going to choose a convenient triangle to finish this off.



$$\mathbb{P}(P \in \triangle ABC) = \left(\frac{x}{x+h}\right)^2$$
$$\mathbb{P}(P, Q, R \in \triangle ABC) = \left(\frac{x}{x+h}\right)^6 = \frac{1}{\left(1+\frac{h}{x}\right)^6} = 1 - \frac{6h}{x} + \cdots$$

Now use Crofton's method. Let $A(x)$ be the expected area of $\triangle PQR$ in isosceles right triangle of side x.

$$A(x+h) = \underbrace{\left(1 - \frac{6h}{x}\right)}_{\mathbb{P}(P,Q,R \in |ABC|)} A(x) + \underbrace{\frac{6h}{x} \cdot \frac{1}{9} \cdot \frac{1}{2} x^2}_{\mathbb{P}(P,Q,\text{ or } R \in \text{ edge})}$$

$$A(x+h) - A(x) + \frac{6h}{x} A(x) = \frac{1}{3} hx$$
$$\frac{A(x+h) - A(x)}{h} + \frac{6A}{x} = \frac{1}{3} x$$

Take limit:

$$\frac{dA}{dx} + \frac{6A}{x} = \frac{1}{3} x$$

Integrating factor is x^6

$$\begin{aligned} x^6 \frac{dA}{dx} + 6Ax^5 &= \frac{1}{3}x^7 \\ \frac{d}{dx}\left(x^6 A\right) &= \frac{1}{3}x^7 \\ x^6 A &= \frac{1}{24}x^8 + C \end{aligned}$$

$A(0) = 0$, so $C = 0$

$$A = \frac{1}{24}x^6 = \frac{1}{12}x^2 2 = \frac{1}{12}|ABC|$$

Thus by the equivalence of these properties under affine transformations, for *any* triangle ABC (not necessarily isosceles right), we have that

$$\mathbb{E}(|PQR|) = \frac{1}{12}|ABC|.$$

Done. Last comment, figure 4.1 in the book. At least try to find the answer when one of the points is out on the edge of the circle using Crofton's method.