Math 191 Notes, 2003 November 25

Geometrical probability

Bertrand's "paradox"

Choose a random chord in unit circle, build an equilateral triangle. Is the triangle inside the circle (i.e. the chord has length less than or equal to $\sqrt{3}$? How to choose a random chord? Many different ways are possible. Not a very cool thing.

- Case 1: Choose a random point in the circle to be the midpoint of the chord. $\mathbb{P}(r \geq \frac{1}{2}) = \frac{3}{4}$.
- Case 2: Choose a random radius, choose a random.point on that radius. $\mathbb{P} = \frac{1}{2}$.
- Case 3: Choose random subtended angle. Triangle fits if

$$|\alpha| \leq \frac{2\pi}{3} \qquad \mathbb{P} = \frac{2}{3}$$

Density functions

$f_{R,\Theta}(p, \theta)$

- Case #1:

$$f(p, \theta) = \underbrace{p}_{\text{Jacobian}} \frac{1}{\pi}$$

- Case #2:
- Case #3:

$$\mathbb{P}(R \leq p) = \mathbb{P}(\cos(\alpha/2) \leq p) = \mathbb{P}(\alpha/2 \geq \arccos p) = 1 - \frac{2}{\pi} \arccos p$$

$$f_R(p) = \frac{2}{\pi\sqrt{1-p^2}} f_{R,\Theta}(p, \theta) = \frac{1}{\pi^2\sqrt{1-p^2}}$$

Buffon's needle

Needle of length L and drop it randomly on a grid of lines of width d. What's the probability that the needle will cross a line?

Poincaré

Drop the plane on the needle. Use a circle with diameter d around the needle. Then exactly one line crosses the circle, and the chord of intersection is uniform on the diameter perpendicular to the stripes.

$$f_{R,\Theta}(p,\theta) = \frac{1}{2\pi}$$

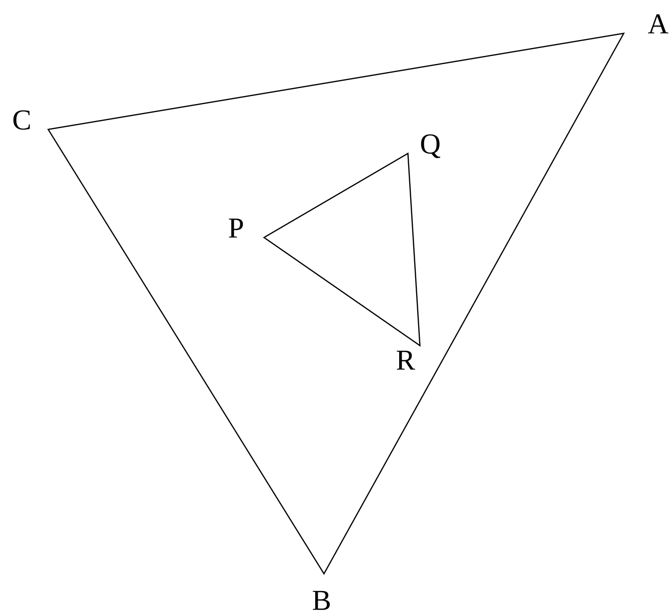
$$P(\text{needle intersects chord}) = \mathbb{P}(|R\sec\theta| \leq L/2)$$

$$= \mathbb{P}(R \leq \frac{|L\cos\theta|}{2} = \frac{|L\cos\theta|}{d}$$

Integrate over θ.

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{|L\cos\theta|}{2} d\theta = \frac{2L}{\pi d}$$

Example



Prove that in any triangle $\triangle ABC$, the random triangle $\triangle PQR$ has expected area of 1/9 of the original. This uses the fact that ratios of areas are preserved under affine transformations.

Lead-in: If X is uniform on $[-l/2, l/2]$, determine $\text{var}(X)$ without integrating. Let $X = Y + Z$

$$Y = \begin{cases} -l/4 & p = 1/2 \\ l/4 & p = -1/2 \end{cases}$$

Z is uniform in $[-l/4, l/4]$. Then,

$$\text{var}(X) = \alpha l^2$$

$$\text{var}(X) = \text{var}(Y) + \text{var}(Z)$$

$$\alpha l^2 = \frac{l^2}{16} + \frac{\alpha}{4} l^2 \Rightarrow \alpha = \frac{1}{12}$$

Lead-in 2: X, Y uniform on $[0, l]$. What is $\mathbb{E}(|X - Y|)$. Most obvious approach is brute for integration.

Crofton's Method. Solve the problem for $[0, z]$ or $[0, z + h]$. The difference is cool. Get a differential equation.