Math 191 Notes, 2003 November 18

Change of variable formulas for double integrals

Let's say we have a plane that represents the values of random variables X, Y and U, V are functions of X, Y. To be precise,

$$U = u(X, Y), \qquad V = v(X, Y)$$

$$X = x(U, V), \qquad Y = y(U, V)$$

where we are stating functions and their inverses. The level curves are the curves such that one of the random variables is constant[1]. Let's evaluate the "parallelogram" inside $v \leq V \leq v + b$ and $u \leq U \leq u + a$ (these do not have to be at right angles). So,

$$\left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}\right) a$$

$$\left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}\right) b$$

A good approximation to the integral here is the area of the parallelogram times the value of the function at the root (or just some point in the parallelogram). Now, let's calculate the density function:

$$\begin{aligned} f_{U,V}(u, v) &= \lim_{a,b \to 0} \frac{(\text{integral of } f_{X,Y}(x, y) \text{ over parallelogram})}{ab} \\ &= \lim_{a,b \to 0} f_{X,Y}(x, y) \left| \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right| \\ &= f_{X,Y}(x(u, v), y(u, v)) |J(u, v)| \end{aligned}$$

3 Example Density Functions

1. Random variables $0 \leq X \leq 1$, and $0 \leq Y \leq 1$ uniform in each.

$$f_{X,Y}(x, y) = \begin{cases} 1 & 0 \leq x, y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

[1] Suppose you are a meteorologist. You could define functions of the air such as temperature $T(x, y, z)$, or density $\rho(x, y, z)$. What do we have to do to change either or both of these functions if we want to change the units? The temperature wouldn't change, but the density function does. Differential 3-forms

2. $x, y \geq 0$, and $x + y \leq 1$, so

$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \leq x+y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

What is the density function for $U = X + Y$?
We need another random variable from which we can compute a Jacobian and go back.
Choice #1: $V = X - Y,\ X = (U+V)/2,\ Y = (U-V)/2$. Jacobian:

$$J(u,v) = \left| \frac{1}{2}\left(-\frac{1}{2}\right) - \frac{1}{2} \cdot \frac{1}{2} \right| = \frac{1}{2}$$

$$f_{U,V}(u,v) = f_{X,Y}(x,y) \cdot \frac{1}{2} = \begin{cases} 1 & 0 \leq u \leq 1,\ -u \leq v \leq u \\ 0 & \text{otherwise} \end{cases}$$

$$f_U(u) = \int_v f_{U,V}(u,v)dv = \int_{-u}^{u} 1dv = 2u$$

This is kind of a lousy choice because the limits of integration are dependent on the value of u. So we choose something that doesn't depend on it: slope.
Choice #2: $V = Y/X,\ X = U/(1+V),\ Y = UV/(1+V)$

$$J(u,v) = \left| \frac{1}{1+v} \frac{-u}{(1+v)^2} - \frac{v}{1+v}\left(\frac{u}{1+v} - \frac{uv}{(1+v)^2}\right) \right| = \frac{u}{(1+v)^2}$$

$$f_{U,V}(u,v) = \frac{2u}{(1+v)^2}$$

$$f_U(u) = \int_0^\infty \frac{2u}{(1+v)^2} dv = 2u$$

U and V are independent here, because the density function is factorable and because we chose it that way.

3. $x, y \geq 0$, and $x^2 + y^2 \leq 1$, so

$$f_{X,Y}(x,y) = \begin{cases} 4/\pi & x^2+y^2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Choose $R = \sqrt{X^2+Y^2},\ \Theta = \arctan \frac{Y}{X}$. These are independent. Let's work out the density function (so we have to invert this). $X = R\cos\Theta,\ Y = R\sin\Theta$.

$$J(r,\theta) = |\cos\theta \cdot r\cos\theta - (r\sin\theta)\sin\theta| = r$$

$$f_{R,\Theta}(r,\theta) = \begin{cases} \frac{4r}{\pi} & 0 \le r \le 1, 0 \le \theta \le \pi/2 \\ 0 & \text{otherwise} \end{cases}$$

We also have a good way to do this with uniform distributions, by going straight from the distribution function in #3.

$$F_{R,\Theta}(r,\theta) = \frac{2}{\pi} r^2 \theta$$

$$f_{R,\Theta}(r,\theta) = \frac{\partial^2}{\partial r \partial \theta}\left(\frac{2}{\pi} r^2 \theta\right) = \frac{4r}{\pi}$$

More generally here are the features that we took advantage of in the above,

1. $f_{X,Y}(x,y) = C\left(\sin^4 x\right)\left(y^3\right)$ is okay because it is independent.
2. $f_{X,Y}(x,y) = C\frac{x^2+xy}{y}$ Change of variables can make this independent $(x+y)\frac{x}{y} = uv$.
3. $x^4 y^3 = r^7 \cos^4\theta \sin^3\theta$