Math 191 Notes, 2003 October 30

Wallis's product

$$
\begin{aligned}
\frac{\pi}{2} &\approx \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdots(2m-2)(2m-2)(2m)}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdots(2m-3)(2m-1)(2m-1)} \\
\sqrt{\frac{\pi}{2}} &\approx \frac{2\cdot 4\cdot 6\cdot 8\cdots(2m-2)\sqrt{2m}}{1\cdot 3\cdot 5\cdot 7\cdots(2m-1)} \\
&= \frac{2\cdot 2\cdot 4\cdot 4\cdots(2m-2)\cdot 2m\cdot\sqrt{2m}}{1\cdot 2\cdot 3\cdot(2m-1)\cdot 2m\cdot 2m} \\
\sqrt{\frac{\pi}{2}} &\approx \frac{2^{2m}(m!)^2\sqrt{2m}}{(2m)!2m} \\
\frac{(2m)!}{2^{2m}(m!)^2} &\approx \frac{1}{\sqrt{\pi m}}
\end{aligned}
$$

Then we can use this to evaluate u_{2m}.

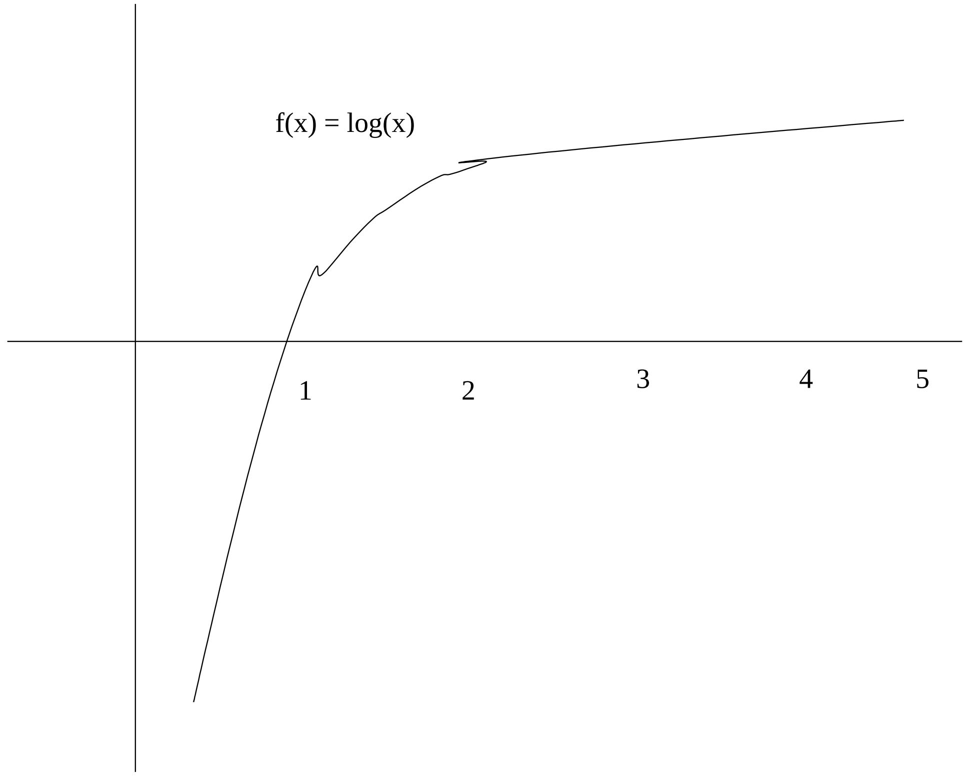
Stirling's Approximation

This is often proved using asymptotic series (very dangerous if we don't know what we're doing). We'll try something half-way there.

$$\log n! = \log 1 + \log 2 + \cdots \log n$$

There is an approximation to this using an integral. using the points, $\frac{1}{2}, \frac{3}{2}, \ldots, n+\frac{1}{2}$.

$$
\begin{aligned}
\int_{1/2}^{n+1/2} \log x\,dx &= \cdots \\
&= (n+1/2)\left(\log n + \frac{1}{2n}\right) - n - \frac{1}{2}\log\frac{1}{2} \\
&= (n+1/2)\log n + 1/2 + 1/4n - n - 1/2\log 1/2 \\
&\quad + \text{unknown Riemann sum error}
\end{aligned}
$$



Let's try $\log n! = (n + 1/2)\log n - n + \log K$

$$
\begin{aligned}
n! &= K n^n e^{-n} \sqrt{n} \\
\frac{(2m)!}{2^{2m}(m!)^2} &\approx \frac{K(2m)^{2m} e^{-2m}\sqrt{2m}}{2^{2m} K^2 m^{2m} e^{-2m} m} \approx \frac{1}{\sqrt{\pi m}} \\
\frac{Km}{\sqrt{2m}} &= \sqrt{\pi m} \\
K &= \sqrt{2\pi} \\
n! &\approx n^n e^{-n} \sqrt{2\pi n}
\end{aligned}
$$

Ok, stirling's approximation is nice. But what about the arcsin? Say we have a sum with a large number of steps, with $0 < x < 1$:

$$
\begin{aligned}
\sum_{k \le xn} u_{2k} u_{2n-2k} &\approx \sum_{k \le xn} \frac{1}{\pi\sqrt{k(n-k)}} \\
&\approx \int_{u=0}^{xn} \frac{du}{\pi\sqrt{u(n-u)}}
\end{aligned}
$$

Set $u = n\sin^2 t$, $du = 2n\sin t\cos t$.

$$\begin{aligned}
\sqrt{u(n-u)} &= \sqrt{n\sin^2 t \cdot n\cos^2 t} = n\sin t\cos t \\
xn &= n\sin^2 t \\
t &= \arcsin\sqrt{x} \\
\int_{u=0}^{xn} \frac{du}{\pi\sqrt{u(n-u)}} &= \int_0^{\arcsin\sqrt{x}} \frac{2dt}{\pi} = \frac{2}{\pi}\arcsin\sqrt{x}
\end{aligned}$$

What does this look like?

$$\frac{2}{\pi}\arcsin\sqrt{x} = \begin{cases} 0 & x = 0 \\ 1 & x = 1 \\ 1/2 & x = 1/2 \\ 1/3 & = 1/4 \end{cases}$$

Has the interesting property that probability of being in the middle half is 1/3. The graph of this is a sine curve on its side.

We are now in continuous probability distributions! We've taken the limiting case of a discrete distribution, and arrived at this.

Continuous distribution functions

$$F(x) = \int_{-\infty}^{x} f(u)du$$

This is the definition—integral of the density function instead of sum of the mass function. Obvious choice of the density function is $f(x) = F'(x)$. Of course, if we have a finite number of removable discontinuites, the integral $F(x)$ need not change.

What about the relevant σ-fields? It's easy to calculate the following probability:

$$\mathbb{P}(a \le X \le b) = \int_a^b f(x)dx$$

Imagine a random variable $Y = g(X)$ given by some periodic graph that oscillates like a sign wave. Then we ask for which x, is Y greater than a certain point. This is a countably infinite union of sets. Let's try the definition:

$$\mathbb{P}(B) = \int_B f(x)dx \text{ where } B \text{ is a union of open intervals}$$

One gotcha: what about the σ-field? The complements of this are closed intervals! Is B in the σ-field? Is there a way to define this? Well, it's named $\mathcal{B}$, it's called the Borel σ-field, and it's the "smallest" σ-field that contains all open intervals.

Expectations

We have a continuous random variable X whose expectation we want to calculate.

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f(x) dx$$

Silly example: X is uniform on the interval $[1, 3]$.

$$\int_1^3 x \cdot \frac{1}{2} dx = \left.\frac{x^2}{4}\right|_1^3 = 2$$

Law of Unconscious Statistician

How about $\mathbb{E}(X^3)$? The idea is that if $Y = g(x)$, then

$$\mathbb{E}(Y) = \int_{-\infty}^{\infty} g(x) f(x) dx$$

. Proof: suppose that $Y = g(X)$ where $g(x) \geq 0$ for all x.
Lemma:

$$\int_0^{\infty} \mathbb{P}(X > y) dy = \int_0^{\infty} dy \int_y^{\infty} f(x) dx = \iint_R f(x) dx dy$$
$$= \int_0^{\infty} dx f(x) \int_0^x dy = \int_0^{\infty} x f(x) dx = \mathbb{E}(X)$$

Proof of unconscious statistician: Let $Y = g(X), g(x) \geq 0$ for all x and $f(x) = 0$ for $x < 0$.

$$\mathbb{E}(Y) = \int_0^{\infty} \mathbb{P}(g(X) > y) dy$$

Find all places where $g(x) > y$, these are the intervals on the graph which are above the horizontal line y, call this $B(y)$. So then we have:

$$\mathbb{E}(Y) = \int_0^{\infty} dy \int_{B(y)} f_X(x) dx = \iint_R f_X(x) dx\, dy$$

$$= \int_0^{\infty} dx f_X(x) - \int_0^{g(x)} dy = \int_0^{\infty} g(x) f_X(x) dx$$