# Math 191 Notes, 2003 October 23

Another reference: Feller, *Introduction to Probability Theory*.

## Random walk questions

Random walk with $p = \frac{1}{2}$ goes through $2n$ steps. Where is it most likely that this crossed zero? What is the distribution of zero-crossing? What is the distribution of the number of times that Harvard was ahead in the series? None of these are very intuitive and the answers you would expect are not true.

## Ballot Theorem

Let's say there is an election in which Arnold gets $n$ votes, and Cruz gets $m$ votes. Let's say (this is not true in real elections) that the votes come in, in random order.

$$\mathbb{P}(\text{Arnold always leads}) = \frac{n-m}{n+m}$$

How to prove this? Approaches: brute force binomial, recurrence relations with boundary conditions, "clever approach," which is an ingenious geometric solution (or "rrbrrrbrbb..."). Say Arnold is eventually going to win the election by $b$ votes, and that he gets the first vote. There are some paths in which Arnold is always ahead. Then there are some paths where $b$ doesn't touch the $x$-axis, and some in which it does. The geometric trick lies in the ability to say that two sets are of the same cardinality because there is a bijection between them. Use the reflection principle like last time.

$$N_{\text{no tie}} = N_{n-1}(1, b) - N_{n-1}(-1, b)$$

Where $n$ is the total number of steps and $b$ at any point in time is Arnold votes minus Cruz votes. Formula from last time:

$$N_m(a, c) = \binom{m}{\frac{1}{2}(m + c - a)}$$

This is the right number of steps for Harvard to have $c - a$ greater wins than Yale. Thus,

$$N_{\text{no tie}} = \binom{n-1}{\frac{n+b}{2} - 1} - \binom{n-1}{\frac{n+b}{2}}$$

Write this out in terms of factorials:

$$\frac{(n-1)!}{\left(\frac{n+b}{2} - 1\right)!\left(\frac{n-b}{2}\right)!} - \frac{(n-1)!}{\left(\frac{n+b}{2}\right)!\left(\frac{n-b}{2} - 1\right)!}$$

Pull out a common factor:

$$\frac{n!}{\left(\frac{n+b}{2}\right)!\left(\frac{n-b}{2}\right)!}\left[\frac{(n+b)/2}{n}-\frac{(n-b)/2}{n}\right]=N_n(0,b)\frac{b}{n}\qquad\square$$

These random walks have many symmetry properties that are useful for physicists, and scientists in general.

## Reversal Principle

Go back to the election, random ordering of the ballots. Rerun the election with the same result but random order. What is the probability that person who achieves the victory does not achieve that margin of victory until the very end? Same thing. Probability of never recrossing the level "b" if we run the random walk backwards from victory and go to zero. $b/n$ again.

$$N_n(0,b)\frac{b}{n}=\begin{cases}\text{\# of paths that never return to level 0}\\ \text{\# of paths that reach level } b \text{ for the first time at step } n\end{cases}$$

Now let's imagine that Harvard and Standford will play every year until the series is tied. Assume $p=\frac{1}{2}$. Also, all Stanford alumni get free tickets when Harvard is exactly $b$ games ahead. What is the expectation $\mu_b$ of the number of free tickets?
Answer: it doesn't matter. $\mu_b=1$ for all $b$. Recall the following:

$$\begin{Bmatrix}\mathbb{P}(S_n=b\text{ with no previous ties})\\ \text{or }\mathbb{P}(S_n=b\text{ for the first time})\end{Bmatrix}=\frac{b}{n}$$

Now, let $f_b(n)$ be defined as follows:

$$f_b(n)=\mathbb{P}(S_n=b\text{ for the first time at step }n)$$

$$=\mathbb{P}(S_n=b\text{ for the first time}|S_n=b)\mathbb{P}(S_n=b)$$

Let's do a numerical example: $b=1$, $n=7$

$$\text{\# sequences }=\binom{7}{3}=35$$

$$\text{\# first time here }=35\cdot\frac{1}{7}=5$$

SSSHHHH, SHSSHHH, SSHSHHH, SHSHSHH, SSHHSHH. Now all we have to do is figure out what $f_b$ is in general.

$f_b(n)$ is defined as the probability of reaching $b$ for the first time after $n$ games. Then let

$$\mu_b = \sum_{n=1}^{\infty} f_b(n)$$

Case $p = \frac{1}{2}$

$$\mu_b = \frac{1}{2}\mu_{b+1} + \frac{1}{2}\mu_{b-1}$$

$$f_b = 1/2 f_{b+1} + 1/2 f_{b-1}$$

Set $b = 0$, $f_0 = 1$, so $B = 1$. $A = 0$, otherwise $f_b \notin [0, 1]$ for some $b$. Argh, muddled.

$$\mu_b = \sum_{n=1}^{\infty} f_b(n) = \sum_{n=1}^{\infty} \mathbb{P}(S_n = b \text{ for the first time})$$

$$f_b(n) = p f_{b-1}(n-1) + q f_{b+1}(n-1)$$

Argh, muddled.

## Key Lemma for arcsin Laws

We're going to state this as probability, but it is actually a counting principle. If $p = \frac{1}{2}$, then

$$u_{2m} = \mathbb{P}(S_1 \neq 0, S_2 \neq 0, \ldots, S_{2m} \neq 0)$$

$$= \mathbb{P}(S_{2m} = 0) = \mathbb{P}(S_1 \geq 0, \ldots, S_{2m} \geq 0)$$

Example: Harvard and Stanford play 4 games.

$$\mathbb{P}(S_4 = 0) = \binom{4}{2}\frac{1}{16} = \frac{3}{8}$$

Remarkable result! This is awesome! Let's see if we can get part of it to work.

$$\begin{aligned} u_2 m &= \underbrace{2}_{\text{H or S}} \sum_{k=1}^{m} \frac{2k}{2m} \mathbb{P}(S_{2m} = 2k) \\ &= 2 \sum_{k=1}^{m} \frac{2k}{2m} \binom{2m}{m+k} \left(\frac{1}{2}\right)^{2m} \end{aligned}$$

Now write $2k = m + k - (m - k)$

$$u_{2m} = \left(\frac{1}{2}\right)^{2m} \cdot 2 \sum_{k=1}^{m} \frac{(2m-1)!}{(m+k)!(m-k)!} [m + k - (m - k)]$$

$$u_{2m} = \left(\frac{1}{2}\right)^{2m} 2\sum_{k=1}^{m} \frac{(2m-1)!}{(m+k-1)!(m-k)!} - \sum_{k=1}^{m} \frac{(2m-1)!}{(m+k)!(m-k-1)!}$$