Math 191 Notes, 2003 October 21

Random walks

Interesting because it is widely applicable. But also because it is uncountably infinite sample space. Nonobvious, counterintuitive, but very simple results.

Examples:

1.

$$X_i = \begin{cases} +1 & \text{prob } p \\ -1 & \text{prob } q = 1 - p \end{cases}$$

$$S_n = a + \sum_{i=1}^{n} X_i$$

This is a "true walk."

2. Harvard-Yale

	W	L	GB
H	54	46	–
Y	46	54	8

This is S_n with $a = 0$.

Properties

- Space invariance: $\mathbb{P}(S_n = j | S_0 = a) = \mathbb{P}(S_n = j + b | S_n = a + b)$

$$= \mathbb{P}\left(\sum_{i=1}^{n} X_i = j - a\right)$$

- Time invariance: $\mathbb{P}(S_n = j | S_0 = a) = \mathbb{P}(S_{n+m} = j | S_m = a)$
- Markov: $\mathbb{P}(S_{m+n} = j | S_0, S_1, \ldots, S_m) = \mathbb{P}(S_{m+n} = j | S_m)$

Absorbing Barriers: "Gambler's Ruin"

If you reach zero, you lose and stop. If you reach N, you win and stop. Question: what is p_k?

$$p_k = \text{ prob. of "ultimate ruin" (reach 0 before reaching N)}$$

Similar to the problem of the points (h heads before t tails).

$$\begin{aligned} p_k &= p(p_{k+1}) + q(p_{k-1}) \\ x^k &= px^{k+1} + qx^{k-1} \\ x &= px^2 + q \\ 0 &= px^2 - x + q \end{aligned}$$

Where we've used the technique for solving second-order linear differential equations. Try the solution x^k, because this is linear. The solutions are $x = 1$ or $x = q/p$. Thus the general solution is:

$$p_k = A_1 \cdot 1 + A_2 \cdot \left(\frac{q}{p}\right)^k$$

Boundary conditions: $p_0 = 1$, $p_N = 0$.

$$\begin{cases} A_1 + A_2 & = & 1 \\ A_1 + A_2(q/p)^N & = & 0 \end{cases}$$

$$A_2 = \frac{1}{1-(q/p)^N}$$

$$A_1 = -\frac{(q/p)^N}{1-(q/p)^n}$$

$$p_k = \frac{(q/p)^k - (q/p)^N}{1-(q/p)^N}$$

provided $p \neq q$, which would have been the most interesting case. If $p = 1/2$, the technique that is used cannot find the section solution (we get a double root of $x = 1$). Standard method for solving differential equations is to multiply by a power of the independent variable.

$$\begin{aligned} p_k &= \frac{1}{2}(p_{k+1} + p_{k-1}) \\ x^k &= \frac{1}{2}(x^{k+1} + x^{k-1}) \\ x &= 1 \end{aligned}$$

So try $p_k = 1 \cdot k$, works! So,

$$p_k = A_1 + A_2 k$$

Where the boundary conditions give us $A_1 = 1$ and $A_2 = \frac{1}{N}$. If $p = 1/2$, $p_k = 1 - k/N$. Thus, as $N \to \infty$, the probability of "ruin" is 1. What is the expected number of steps before "ruin"?

$$D_k = \text{ expected steps before ruin starting at level } k$$

$$D_k = p \cdot (1 + D_{k+1}) + q \cdot D(1 + D_{k-1}) = 1 + pD_{k+1} + qD_{k-1}$$

Specialize to $p = 1/2$.

$$D_k = 1 + \frac{1}{2}(D_{k+1} + D_{k-1})$$

Try the solution $-k^2$

$$-k^2 = 1 + \frac{1}{2}\left(-(k+1)^2 - (k-1)^2\right)$$

General solution:

$$D_k = -k^2 + A_1 + A_2 k$$

Boundary condition. $D_0 = 0, D_N = 0$

$$D_k = -k^2 + Nk = k(N-k)$$

As $N \to \infty$, then $D_k \to \infty$.

Ballot Theorem

Harvard and Yale play 5 games, and Harvard wins 3. For what fraction of the possible ways is the series never tied. Answer: $\frac{3-2}{5} = \frac{1}{5}$
Number of possible ways: $\binom{5}{2} = 10$
Random walk equivalent: $\frac{b}{n}$.
Physics example: reflection principle. The shortest path that light takes from a to b reflecting through a mirror is the same as if we just reflected b across the mirror and connected a and b with a straight line. Use this kind of thing to prove the Ballot Theorem.

If $a, b > 0$, Let $N_n^0(a,b)$ = number of paths from a to b for which $S_k = 0$ for some k. By reflection, $N_n^0(a,b) = N_n(-a,b)$.

Say there are H Harvard wins in going from a to b in n steps. The answer is $a + H - (n - H) = b \Rightarrow H = 1/2(n + b - a)$.

$$N_n(a,b) = \binom{n}{\frac{1}{2}(n+b-a)}$$

Hmm, I don't think that's correct. Maybe that should just be a fraction and not a binomial coefficient.