Math 191 Notes, 2003 October

Indpendence?

Independent X,Y means

$$\begin{aligned}\mathbb{P}(X \le x, Y \le y) &= \mathbb{P}(X \le x)\mathbb{P}(Y \le y) \\ \mathbb{P}(X = x, Y = y) &= \mathbb{P}(X = x)\mathbb{P}(Y = y)\end{aligned}$$

Example: Toss N coins (Poisson) get x heads, y tails. X = number of heads, Y = number of tails.

$$\begin{aligned}\mathbb{P}(X = x, Y = y) &= e^{-\lambda}\frac{\lambda^{(x+y)}}{(x+y)!}\frac{(x+y)!}{x!y!}p^x q^y \\ &= \left(\frac{e^{-p\lambda}\lambda^x p^x}{x!}\right)\left(\frac{e^{-q\lambda}\lambda^y q^y}{y!}\right)\end{aligned}$$

If $f(k) = e^{-\lambda}\frac{\lambda^k}{k!}$ for some λ, "X has a Poisson distribution."

Cauchy-Schwarz Inequality

Take random variables X and Y, look at $aX+bY$. That is a vector space of random variables. Is tehre any way we could assign a non-negative inner product whose square root might be interpreted as a length? Look at variance. Where $\mathbb{E}(X) = 0$, $\mathbb{E}(X^2)$, $\mathbb{E}(X)\mathbb{E}(Y) = \mathbb{E}(XY)$. Let $Z = \lambda X + Y$.

$$\mathbb{E}(Z^2) = \mathbb{E}(\lambda^2 X^2 + 2\lambda XY + Y^2) \ge 0$$

$$\lambda^2\mathbb{E}(X^2) + 2\lambda\mathbb{E}(XY) + E(Y^2) \le 0$$

Unles $Z = 0$, this quadratic has no real roots. Therefore,

$$(2\mathbb{E}(XY))^2 - 4\mathbb{E}(X^2)\mathbb{E}(Y^2) \le 0$$

$$\mathbb{E}(XY) \le \sqrt{\mathbb{E}(X^2)\mathbb{E}(Y^2)}$$

Now use $X - \mathbb{E}(X)$ and $Y - \mathbb{E}(Y)$

$$\mathbb{E}((X - \mathbb{E}(X))(Y - \mathbb{E}(Y))) \le \sqrt{\mathbb{E}((X - \mathbb{E}(X))^2)\mathbb{E}((Y - E(Y))^2)}$$

$$cov(X, Y) \le var(X0var(Y)$$

Given X, Y with expecation 0. We call the correlation, ρ as

$$\rho(X, Y) \equiv \frac{cov(X, Y)}{\sqrt{var(X)var(Y)}}$$

$$|\rho(X,Y)| \leq 1$$

with equality only if $\lambda X + Y = 0$.

Poisson

Random N (number of apps) is Poisson with parameter λ. Each application is successful independently with probability p.

- K is a random variable, the number of acceptances.
- Average number of applications $\mathbb{E}(N) = \lambda$
- Average number of acceptances $p\lambda$
- $\mathbb{E}(K|N=n) = pn$
- E(N — K=k)

Now,

$$\begin{aligned}
\mathbb{P}(N=n|K=k) &= \frac{\mathbb{P}(N=n, K=k)}{\mathbb{P}(K=k)} \\
&= \frac{\mathbb{P}(K=k|N=n)\mathbb{P}(N=n)}{\mathbb{P}(K=k)} \\
\mathbb{P}(N=n|K=k) &= \frac{\binom{n}{k} p^k q^{n-k} e^{-\lambda} \frac{\lambda^n}{n!}}{\sum_{n\geq k} \binom{n}{k} p^k q^{n-k} e^{-\lambda} \frac{\lambda^n}{n!}}
\end{aligned}$$

Then, setting $n - k = i$,

$$\begin{aligned}
\mathbb{E}(N|K=k) &= \sum n \mathbb{P}(N|K=k) \\
&= \frac{k \sum \frac{(q\lambda)^i}{i!} + \sum i \frac{(q\lambda)^i}{i!}}{\sum \frac{(q\lambda)^i}{i!}} = k + q\lambda
\end{aligned}$$