

# Math 191 Notes, 2003 October 14

## Matching Problem, Review of specific case

Matching problem, general case with  $n$  letters. Let the probability that the  $r = 0$  (number of correct letters is zero).

$$PP(r = 0) = \frac{1}{n!} \sum_{\pi} (1 - I_1) \cdots (1 - I_n)$$

For any  $n$ , the probability that any specific set of  $n$  letters end up in their correct envelopes is the same as the probability for any other set of  $n$  letters.

$$PP(r = 0) = \frac{1}{n!} \sum_s (-1)^s \binom{n}{s} (n-s)! I_1 \cdots I_s$$

Thus we get that general formula from last time.

## Matching Problem, General case

Let  $X$  be the random variable defining the number of “good” letters. What is  $PP(X = r)$ ? Pick the  $r$  that will be correct. Then make all the the rest  $(n - r)$  incorrect.

$$PP(X = r) = \frac{1}{n!} (n-r)! \sum_{s=0}^{n-r} (-1)^s \frac{1}{s!} = \frac{1}{r!} \sum_{s=0}^{n-r} \frac{(-1)^s}{s!}$$

As  $n \rightarrow \infty$ , this is a poisson distribution with  $\lambda = 1$ . This is one of many ways of getting that distribution.

## “Probabilistic method”

Say we have 17 fenceposts in a circle. 5 of them are rotten. Prove there is at least one set of seven consecutive posts of which 3 are rotten.

Solution using probabilistic concepts: Let  $R_k$  be the number of rotten posts in  $\{k+1, k+2, \dots, k+7 \pmod{17}\}$ .

$$EE(R_k) = \sum_{i=1}^7 I_{k+i} = 7 \cdot \frac{5}{17} = \frac{35}{17} > 2$$

Where  $I_k$  is the indicator function for post  $k$  being rotten. Since the probability is greater than two, we know there must be event that has three fence posts.

## Binomial $\rightarrow$ Poisson

First we want to prove that Poisson is limiting case of Binomial distribution. Binomial distribution:

$$f(k) = \binom{n}{k} p^k q^{n-k} = \frac{n!}{(n-k)!k!} \left(\frac{p}{q}\right)^k q^n$$

Now let  $p \rightarrow 0, n \rightarrow \infty$  hold  $np = \lambda$ .

$$f(k) = \left(\frac{np^k}{q}\right) \frac{1}{k!} \lim_{n \rightarrow \infty} \left(1 - \frac{\lambda}{n}\right)^n = \frac{\lambda^k}{k!} e^{-\lambda}$$

Aside: for any specified value of  $k$ , we can probably do what we were hoping and take the limit as  $n \rightarrow \infty$ . Also, here's a differentiation trick.

$$\int_0^\infty e^{-kx} dx = \frac{1}{k}$$

$$\begin{aligned} \int_0^\infty x^2 e^{-kx} dx &= \int_0^\infty \frac{d^2}{dk^2} e^{-kx} dx \\ &= \frac{d^2}{dk^2} \left(\frac{1}{k}\right) \\ &= \frac{2}{k^3} \end{aligned}$$

## Expectation for binomial distribution

$$\begin{aligned} EE(k) = \sum_{k=0}^n k f(k) &= \sum k \binom{n}{k} p^k q^{n-k} \\ &= \sum_{k=0}^n p \frac{\partial}{\partial p} \left[ \binom{n}{k} p^k q^{n-k} \right] \\ &= p \frac{\partial}{\partial p} \sum \binom{n}{k} p^k q^{n-k} \\ &= p \frac{\partial}{\partial p} (p+q)^n \\ &= pn(p+q)^{n-1} = np \end{aligned}$$

And take the limit to get Poisson  $np = \lambda$ .

## Variance for binomial distribution

$$\text{var}(X) = EE(X^2) - (E(X))^2$$

$$\begin{aligned}
EE(X^2) &= \sum_{k=0}^n k^2 f(k) \\
&= \sum_{k=0}^n k^2 \binom{n}{k} p^k q^{n-k} \\
&= p \partial / \partial p \left( p \partial / \partial p \sum_{k=0}^n \binom{n}{k} p^k q^{n-k} \right) \\
&= p \partial / \partial p p \partial / \partial p (p+q)^n \\
&= p \partial / \partial p (pn(p+q)^{n-1}) \\
&= p [n(p+q)^{n-1} + pn(n-1)(p+q)^{n-2}] \\
&= np + n(n-1)p^2
\end{aligned}$$

$$var(X) = np + n^2 p^2 - np^2 - n^2 p^2 = np(1-p) = npq$$

For the Poisson, we start with binomial and  $np \rightarrow \lambda, p \rightarrow 0$ :  $var(X) = \lambda$ .

## Geometric Distribution

“Geometric” distribution: How many flips  $k$  do I need to see the first head?

$$f(k) = q^{k-1}p$$

$$\sum_1^\infty f(k) = p \sum_{k=1}^\infty q^{k-1} = \frac{p}{1-q} = 1$$

Expectation:

$$EE(X) = \sum_1^\infty k f(k) = p \sum_{k=1}^\infty k q^{k-1} = p \frac{d}{dq} \left( \sum_{k=0}^\infty q^k \right) = p \frac{d}{dq} \frac{1}{1-q} = p \frac{1}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$$

Variance:

$$\begin{aligned}
\sum_{k=1}^\infty k^2 f(k) &= p \sum_{k=1}^\infty k^2 q^{k-1} = p \sum_{k=1}^\infty \frac{d}{dq} \left( q \frac{d}{dq} q^k \right) = p \frac{d}{dq} \left( q \frac{d}{dq} \frac{1}{1-q} \right) = p \frac{d}{dq} \left( q \frac{1}{(1-q)^2} \right) = p \left[ \frac{1}{(1-q)^2} + \frac{2q}{(1-q)^3} \right] \\
var(X) &= \frac{1}{p} + \frac{2(1-p)}{p^2} - \frac{1}{p^2} = \frac{1}{p^2} - \frac{1}{p} = \frac{1-p}{p^2} = \frac{q}{p^2}
\end{aligned}$$

## Negative Binomial Distribution

How many times does it take to get  $r$  6's when rolling a die?