DIRICHLET'S THEOREM ON PRIMES IN ARITHMETIC PROGRESSIONS

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1. Introduction

It is a well-known fact that there are infinitely many primes. However, it is less clear how the primes are distributed throughout the integers. A natural question to ask is whether there are infinitely many primes in arithmetic progressions: sequences of the form $a, a+q, a+2q, a+3q, \ldots$. Clearly, there will be none in some sequences. If $a = 4$ and $q = 8$ then all elements of the sequence will be divisible by 4, so there clearly can't be any primes. In other progressions it is less clear. If $a = 5$ and $q = 7$, then there are many primes: the first 7 elements of the sequence are $5, 12, 19, 26, 33, 40, 47$, which already contains 3 primes. It turns out that if no number other than 1 divides all elements of the sequence, there will be infinitely many primes in the sequence. More formally,

Theorem 1. *If a, q are positive integers with $(a, q) = 1$, then there are infinitely many primes in the sequence $\{a + np\}_{n\in\mathbb{N}}$.*

We seek to prove this theorem. To do so, we first introduce two objects, the Dirichlet series $A(s, f)$ and the Dirichlet character χ_q, and prove some useful results about them. We will then use these to show that

$$\lim_{s\to 1} \sum_{p\equiv a \pmod q} \frac{1}{p^s} = \infty$$

which will show that there must be infinitely many primes in the sum on the left.

We assume some basic knowledge of number theory (the properties of the group of units modulo n) and complex analysis (some familiarity with analytic functions).

2. Dirichlet Series

Definition 2. A *Dirichlet series* is a sum of the form

$$A(s, f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}, \tag{1}$$

The *Riemann-Zeta Function*, $\zeta(s)$, is the Dirichlet series where $f(n) = 1$ for all n.

The question of convergence for Dirichlet series is a complicated one. However, if we restrict our attention to absolute convergence, we can quickly obtain a useful result:

Theorem 3. *For each Dirichlet series* $A(s,f)=\sum_{n=1}^{\infty} f(n)/n^s$ *there exists a unique number* $\sigma_f \in \mathbb{R}\cup\{\pm\infty\}$ *such that the series* $A(s,f)$ *is absolutely convergent for* $\operatorname{Re} s > \sigma_f$*, and not for* $\operatorname{Re} s < \sigma_f$*.*

Proof. We follow the proof in [1].

Let S be the set of all α such that

$$\sum_{n=1}^{\infty} |f(n)| n^{-\alpha} < \infty.$$

Since $|n^{-s}| = n^{-\operatorname{Re} s}$ is a monotonically decreasing function of $\operatorname{Re} s$, we know from the comparison test that the series $\sum_{n=1}^{\infty} f(n)n^{-s}$ is absolutely convergent for $\operatorname{Re} s > \alpha$; thus if $\alpha \in S$ then all $\beta > \alpha$ are in S. Let $\sigma_f = \inf S$. Then by the definition of σ_f, we know that there does not exist s with $\operatorname{Re} s < \sigma_f$ such that $\sum_{n=1}^{\infty} f(n)n^{-s}$ is absolutely convergent.

So we have shown that if S is nonempty we are done. If S is empty, set $\sigma_f = \infty$, and the theorem will hold for σ_f. $\square$

Notice that if we apply the integral test to the series for $\zeta(s)$, we get that $\sigma_f = 1$.

If we make some extra assumptions on f we can obtain a useful factorization of the series, known as the *Euler product*. The existence of such a factorization hints at the connection between these series and theorems about primes; it is often useful to take the logarithm of such a product to obtain facts about sums over primes.

Proposition 4. *If* $A(s,f)=\sum_{n=1}^{\infty} f(n)n^{-s}$ *and* $f(nm)=f(n)f(m)$ *for all* n,m *then we have*

$$A(s,f) = \prod_p \left(1-\frac{f(p)}{p^s}\right)^{-1} \tag{2}$$

for all s *such that* $\operatorname{Re} s > \sigma_f$*.*

Proof. Notice that for $\operatorname{Re} s > \sigma_f$

$$\sum_{k=0}^{\infty} \frac{|f(p^k)|}{|p^{sk}|} \le \sum_{n=1}^{\infty} \frac{|f(n)|}{|n^s|},$$

so the series on the left converges, and thus equals $(1-f(p)/p^s)^{-1}$. Also, notice that for the first N primes $p_1,\ldots,p_N$, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{f(n)}{n^s} - \prod_{m=1}^{N}\left(1-\frac{f(p_m)}{p_m^s}\right)^{-1} &= \sum_{n=1}^{\infty}\frac{f(n)}{n^s} - \sum_{e_1,\ldots,e_N}\frac{f(p_1^{e_1}\cdots p_N^{e_N})}{p_1^{e_1 s}\cdots p_N^{e_N}} \\ &\le \sum_{n=p_N+1}^{\infty}\frac{f(n)}{n^s}\end{aligned}$$

which can be made arbitrarily small since $A(s,f)$ exists. Thus the two formulas converge to the same value, and we are done. $\square$

The Riemann Zeta function is the most famous Dirichlet series. It is useful in proving theorems about the distributions of primes. Here we prove a simple bound on $\zeta(s)$:

Proposition 5. *For real $s > 1$,*

$$\zeta(s) = \frac{1}{s-1} + O(1)$$

uniformly.

Proof. Notice that $(n+1)^{-s} \leq \int_n^{n+1} u^{-s}\,du \leq n^{-s}$ for $n \in \mathbb{N}$. Summing over n we see that $\zeta(s) - 1 \leq \int_1^{\infty} u^{-s}\,du \leq \zeta(s)$ for $s > 1$. Computing the integral and rearranging the inequality, we find that

$$\frac{1}{s-1} \leq \zeta(s) \leq \frac{1}{s-1} + 1$$

for $s > 1$. Thus

$$\zeta(s) = \frac{1}{s-1} + O(1).$$

□

3. Dirichlet Characters

Definition 6. A *Dirichlet character* is a function $\chi_q : \mathbb{Z} \to \mathbb{C}$ such that

(1) if $n \equiv m \pmod{q}$ then $\chi_q(m) = \chi_q(n)$,
(2) $\chi_q(mn) = \chi_q(m)\chi_q(n)$ for all $m, n \in \mathbb{Z}$, and
(3) $\chi_q(n) = 0$ if and only if $(n, q) \neq 1$.

Although the definition of Dirichlet characters does not seem restrictive, it turns out that it is very simple to classify all Dirichlet characters. First, in order to simplify our exposition, we introduce some notation:

Definition 7. We denote the cardinality of of $(\mathbb{Z}/n\mathbb{Z})^*$ by $\varphi(n)$. For any integer a such that $(a, n) = 1$ we denote the smallest integer k such that $a^k \equiv 1 \pmod{n}$ by $\mathrm{ord}_n(a)$.

Definition 8. We denote the complex number $\exp(2\pi/a)$, a primitive a-th root of unity, by ζ_a.

Proposition 9. *There are exactly $\varphi(q)$ distinct χ_q. If $b_1, \ldots, b_m$ are the generators of $(\mathbb{Z}/q\mathbb{Z})^*$ then the χ_q are exactly the functions defined by*

$$\chi_q(a) = \zeta_{\mathrm{ord}_q(b_1)}^{k_1 e_1} \cdots \zeta_{\mathrm{ord}_q(b_m)}^{k_m e_m}$$

where $0 \leq k_j < \mathrm{ord}_q(b_j)$ if $a \equiv b_1^{e_1} \cdots b_m^{e_m} \pmod{q}$ and $\chi_q(a) = 0$ otherwise.

We will denote the character where $k_j = 0$ for all $j = 1, \ldots, m$ by ϵ_q.

Proof. Plugging $n = m = 1$ into property 2 we see that $\chi_q(1 \cdot 1) = \chi_q(1)^2$ so $\chi_q(1) = 0$ or 1. Since it can't equal 0 by property 3, $\chi_q(1) = 1$. Suppose that $(a, q) = 1$. Then $a^{\varphi(q)} \equiv 1 \pmod q$, so $\chi_q(a)^{\varphi(q)} = \chi_q(1) = 1$, so $\chi_q(a)$ must be a $\varphi(q)$-th root of 1.

Notice that since $\chi_q(b_j^{\mathrm{ord}_q(b_j)}) = 1$, we know that $\chi_q(b_j) = \zeta_{ord_q(b_j)}^{k_j}$ for some $0 \le k_j < \mathrm{ord}_q(b_j)$. Thus all χ_q must be of the desired form. Now fix k_j for each b_j, and define χ_q on the rest of $(\mathbb{Z}/q\mathbb{Z})^*$ by $\chi_q(b_1^{e_1} \cdots b_m^{e_m}) = \chi_q(b_1)^{k_1 e_1} \cdots \chi_q(b_m)^{k_m e_m}$. Define $\chi_q(a) = 0$ if $(a, q) \neq 1$. Then by definition this function will satisfy the definition of a Dirichlet character. Thus all Dirichlet characters satisfy the given formulas.

Notice that from the formula we see that there are $\prod_{k=1}^m \mathrm{ord}_q(b_j)$ different Dirichlet characters. But $\prod_{j=1}^m \mathrm{ord}_q(b_j) = \#(\mathbb{Z}/q\mathbb{Z})^* = \varphi(q)$ by definition. □

Theorem 10. *For* $(a, q) = 1$,

$$\sum_{\chi_q} \chi_q(a)^{-1} \chi_q(p) = \begin{cases} \varphi(q) & \text{if } a \equiv p \pmod q \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $b_1, \ldots, b_m$ be generators of $(\mathbb{Z}/q\mathbb{Z})^*$. If $(p, q) \neq 1$ then the theorem is trivially true. So now suppose that $(p, q) = 1$. Let $a = \prod_{j=1}^m b_j^{e_j}$ and let $p = \prod_{j=1}^m b_j^{f_j}$. Then

$$\begin{aligned} \sum_{\chi_q} \chi_q(a)^{-1} \chi_q(p) &= \sum_{\chi_q} \left(\prod_{j=1}^m \chi_q(b_j)^{-e_j} \right) \left(\prod_{j=1}^m \chi_q(b_j)^{f_j} \right) \\ &= \sum_{\chi_q} \prod_{j=1}^m \chi_q(b_j)^{f_j - e_j} \end{aligned}$$

If $f_j = e_j$ (so $p \equiv a \pmod q$) this is just $\sum_{\chi_q} 1 = \varphi(q)$. Otherwise, suppose that m' is the largest index such that $f_{m'} \neq e_{m'}$. Then if we let $C = \prod_{j=m'+1}^m \mathrm{ord}_q(b_j)$ then

$$\begin{aligned} \sum_{\chi_q} \prod_{j=1}^m \chi_q(b_j)^{f_j - e_j} &= C \sum_{k_1=1}^{\mathrm{ord}_q(b_1)} \cdots \sum_{k_{m'}=1}^{\mathrm{ord}_q(b_{m'})} \prod_{j=1}^{m'} \zeta_{\mathrm{ord}_q(b_j)}^{k_j(f_j - e_j)} \\ &= C \sum_{k_1=1}^{\mathrm{ord}_q(b_1)} \cdots \sum_{k_{m'-1}=1}^{\mathrm{ord}_q(b_{m'-1})} \prod_{j=1}^{m'-1} \zeta_{\mathrm{ord}_q(b_j)}^{k_j(f_j - e_j)} \sum_{k_{m'}=1}^{\mathrm{ord}_q(b_{m'})} \zeta_{\mathrm{ord}_q(b_{m'})}^{k_{m'}(f_{m'} - e_{m'})} \\ &= C \sum_{k_1=1}^{\mathrm{ord}_q(b_1)} \cdots \sum_{k_{m'-1}=1}^{\mathrm{ord}_q(b_{m'-1})} \prod_{j=1}^{m'-1} \zeta_{\mathrm{ord}_q(b_j)}^{k_j(f_j - e_j)} \frac{1 - \zeta_{\mathrm{ord}_q(b_{m'})}^{(f_{m'} - e_{m'})\mathrm{ord}_q(b_{m'})}}{1 - \zeta_{\mathrm{ord}_q(b_{m'})}^{f_{m'} - e_{m'}}} \\ &= 0 \end{aligned}$$

because $\zeta_{\mathrm{ord}_q(b_{m'})}^{(f_{m'} - e_{m'})\mathrm{ord}_q(b_{m'})} = 1$. So we are done. □

4. Proof of the Main Theorem

Theorem 10 tells us that we can pick out one equivalence class modulo q in a sum of Dirichlet series by summing over $A(s, \chi_q)$ with the correct coefficients. More specifically, we will take the logarithm of the Euler product decompositions of $A(s, \chi_q)$ and take a linear sum of those. Since the sums in the logarithms will be over primes p, we will be able to make a sum that is over only primes that are equivalent to a fixed a modulo q. All that will remain will be to show that those sums are infinite.

Notice that by comparison with ζ, we see that σ_f for $A(s, \chi_q)$ is at most 1. We show the connection between $\log A(s, \chi_q)$ and an infinite sum over primes:

Theorem 11. *Let χ_q be a Dirichlet character. Then for real $s > 1$,*

$$\sum_p \sum_{k=1}^{\infty} \frac{1}{k} \chi_q(p)^k p^{-ks} = \log A(s, \chi_q).$$

Proof. We follow the proof in [1].

Take logarithms of both sides of (2). Then we find that

$$\log A(s, \chi_q) = \sum_p \log(1 - \chi_q(p) p^{-s})^{-1}.$$

Since $|\chi_q(p) p^{-s}| < 1$, we can use the power series expansion for $\log(1-z)^{-1}$ to get that

$$\log A(s, \chi_q) = \sum_p \sum_{k=1}^{\infty} \frac{1}{k} \chi_q(p)^k p^{-ks}.$$

□

(As in [1]) So we now have an infinite sum over prime powers. To reduce this to a sum over primes, we note that for $s \geq 1$

$$\sum_p \sum_{k=2}^{\infty} \frac{1}{k} p^{-ks} \leq \sum_p \sum_{k=2}^{\infty} p^{-ks} = \sum_p \frac{p^{-2s}}{1 - p^{-s}} \leq \sum_p \frac{1}{p(p-1)}$$

which converges by comparison with $1/n^2$. Thus from the above we see that

$$\sum_p \frac{\chi_q(p)}{p^s} = \log A(s, \chi_q) + O(1).$$

Then for integers a, q such that $(a, q) = 1$ and $s > 1$

(3)
$$\frac{1}{\varphi(q)} \sum_{\chi_q} \chi_q(a)^{-1} \log A(s, \chi_q) = \frac{1}{\varphi(q)} \sum_p \sum_{\chi_q} \frac{\chi_q(p) \chi_q(a)^{-1}}{p^s} + O(1) = \sum_{p \equiv a \pmod{q}} \frac{1}{p^s} + O(1).$$

So we have now written a sum over primes equivalent to a modulo q. It remains to give an estimate of this sum.

First, we give an asymptotic for $A(s, \epsilon_q)$, which has a pole at $s = 1$:

Proposition 12.

$$A(s,\epsilon_q)=\frac{1}{s-1}+O(q).$$

Proof. Notice that for $s>1$

$$A(s,\epsilon_q)=\zeta(s)-\prod_{p|q}(1-p^{-s})^{-1}.$$

So if we could show that $\prod_{p|q}(1-p^{-s})^{-1}$ is $O(q)$, we would be done. Indeed,

$$\prod_{p|q}(1-p^{-s})^{-1}\leq\prod_{p|q}(1-2^{-s})^{-1}\leq\prod_{p|q}2\leq q=O(q).$$

So we are done. □

Now we can estimate the Dirichlet series for the other characters.

Proposition 13. *If χ_q is a Dirichlet character not equal to ϵ_q, then $A(s,\chi_q)$ is a continuous function whenever* $\operatorname{Re} s>0$.

Proof. We follow [3]. Let $S(x)=\sum_{n\leq x}\chi_q(n)$.

Notice that if χ_q is a Dirichlet character not equal to ϵ_q, we know that

$$\begin{aligned}\sum_{k=\ell}^{\ell+q}\chi_q(k) &= \sum_{k=1}^{q}\chi_q(k)=\sum_{e_1=1}^{\operatorname{ord}_q(b_1)}\cdots\sum_{e_m=1}^{\operatorname{ord}_q(b_m)}\chi_q(b_1^{e_1}\cdots b_m^{e_m})\\ &= \sum_{e_1=1}^{\operatorname{ord}_q(b_1)}\cdots\sum_{e_{m-1}=1}^{\operatorname{ord}_q(b_{m-1})}\chi_q(b_1^{e_1}\cdots b_{m-1}^{e_{m-1}})\sum_{e_m=1}^{\operatorname{ord}_q(b_m)}\chi_q(b_m^{e_m})\\ &= \sum_{e_1=1}^{\operatorname{ord}_q(b_1)}\cdots\sum_{e_{m-1}=1}^{\operatorname{ord}_q(b_{m-1})}\chi_q(b_1^{e_1}\cdots b_{m-1}^{e_{m-1}})\frac{1-\chi_q(b_m)^{\operatorname{ord}_q(b_m)}}{1-\chi_q(b_m)}=0.\end{aligned}$$

so $|S(x)|$ is uniformly bounded by $\max_{1\leq k<q}\left|\sum_{j=1}^{k}\chi_q(j)\right|$.

Now observe that

$$\begin{aligned}A(s,\chi_q) &= \sum_{n=1}^{\infty}S(n)\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right)\\ &= s\sum_{n=1}^{\infty}S(n)\int_n^{n+1}x^{-s-1}\,dx=s\int_1^{\infty}S(x)x^{-s-1}.\end{aligned}$$

Since $S(x)$ is uniformly bounded the above integral is analytic and converges for all s with $\operatorname{Re} s>0$, so $A(s,\chi_q)$ is an analytic function on the half-plane $\operatorname{Re} s>0$. □

In particular, this means that $A(s,\chi_q)$ tends to a finite limit as $s \to 1$. If we could show that $A(1,\chi_q) \neq 0$ then $\log A(1,\chi_q)$ would be finite, and by considering (3) we would know that

$$\sum_{p \equiv a \pmod{q}} \frac{1}{p^s} = \frac{1}{\varphi(q)} \log \frac{1}{s-1} + O(\log q).$$

This tends to infinity as $s \to 1$, so there must be infinitely many primes in the left-hand sum. This is Dirichlet's theorem:

Theorem 14. *If a, q are positive integers such that $(a,q) = 1$ then there are infinitely many primes in the sequence $\{a + kq\}_{k \in \mathbb{N}}$.*

We finish the proof by showing the following:

Theorem 15. *If χ_q is a Dirichlet character not equal to ϵ_q, then $A(1,\chi_q) \neq 0$.*

Proof. We follow [3].

First we will prove a lemma:

Lemma 15.1. *For real $s > 0$,*

$$\prod_{\chi_q} A(s,\chi_q) \geq 1.$$

Proof. Recall that

$$\log A(s,\chi_q) = \sum_p \sum_{k=1}^{\infty} \frac{1}{k} \chi_q(p^k) p^{-ks}.$$

Summing over χ_q and using theorem 10 we see that

$$\sum_{\chi_q} \log A(s,\chi_q) = \sum_p \sum_{k=1}^{\infty} \frac{1}{k} \left(\sum_{\chi_q} \chi_q(p^k) \right) p^{-ks} = \varphi(q) \sum_{p^k \equiv 1 \pmod{m}}^{\infty} \frac{1}{k} p^{-ks}.$$

For real s the right-hand side is nonnegative. Thus

$$\prod_{\chi_q} A(s,\chi_q) \geq 1.$$

$\square$

Now suppose that χ_q is a complex character (so there exists some $n \in \mathbb{Z}$ such that $\chi_q(n) \notin \mathbb{R}$). From the definition of $A(s,\chi_q)$ we see that $\overline{A(s,\chi_q)} = A(\overline{s},\overline{\chi}_q)$. Thus if $A(1,\chi_q) = 0$ then $A(1,\overline{\chi}_q) = 0$. In proposition 12 we saw that $A(s,\epsilon_q)$ had a simple pole at $s = 1$; $A(s,\chi)$ is analytic in the right half-plane for $\chi \neq \epsilon_q$. Thus if $A(1,\chi_q)$ has a zero at $s = 1$ so does $A(1,\overline{\chi}_q)$, and so it will have a double zero but only a single pole at $s = 1$, so the product will also be zero. However, from lemma 15.1 we saw that the product is at least 1: contradiction. So for all complex characters χ_q, $A(1,\chi_q) \neq 0$.

So now we simply need to prove the statement for real χ_q, meaning $\chi_q(n) = \pm 1$ for all n relatively prime to q. To facilitate this, we follow the exposition of a proof due to de la Vallée Poussin in [3] and prove the following lemma:

Lemma 15.2. *Suppose f is a nonnegative function on $\mathbb{N}$ such that if $(m, n) = 1$ then $f(mn) = f(m)f(n)$. Suppose also that there is a constant c such that $f(p^k) < c$ for all prime powers p^k. Then $\sum_{n=1}^{\infty} f(n)n^{-s}$ converges for all real $s > 1$, and*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left(1 + \sum_{k=1}^{\infty} \frac{f(p^k)}{p^{ks}}\right).$$

Proof. Fix $s > 1$, and let $a(p) = \sum_{k=1}^{\infty} f(p^k)p^{-ks}$. Then

$$a(p) < c \sum_{k=1}^{\infty} \frac{1}{p^{ks}} = \frac{cp^{-s}}{1 - p^{-s}} < 2cp^{-s}.$$

Thus, using the fact that $1 + x < e^x$ for $x > 0$ we have

$$\prod_{p \leq N} (1 + a(p)) < \prod_{p \leq N} e^{a(p)} = \exp \sum_{p \leq N} a(p).$$

But we know that $\sum_{p \leq N} a(p) < 2c \sum_p p^{-s} = M$. But from the multiplicativity of f we see that

$$\sum_{n=1}^{N} f(n)n^{-s} < \prod_{p \leq N} (1 + a(p)) < \exp M$$

for all N. Since f is nonnegative, we see that $\sum_{n=1}^{\infty} f(n)n^{-s}$ converges.

The last part follows from applying the same reasoning as in proposition 4 to this function. □

Now suppose that χ_q is a real character and that $A(1, \chi_q) = 0$. Consider

$$\psi(s) = \frac{A(s, \chi_q)A(s, \epsilon_q)}{A(2s, \epsilon_q)}.$$

The zero of $A(s, \chi_q)$ cancels out the pole of $A(s, \epsilon_q)$, so the numerator is analytic for $\operatorname{Re} s > 0$. The denominator is analytic for $\operatorname{Re} s > 1/2$, so ψ is analytic for $\operatorname{Re} s > 1/2$. The denominator has a pole at $s = 1/2$, so as $s \to 1/2$, $\psi(s) \to 0$.

Suppose that s is real and $s > 1/2$. Then

$$\psi(s) = \prod_p (1 - \chi_q(p)p^{-s})^{-1}(1 - \epsilon_q(p)p^{-s})^{-1}(1 - \epsilon_q(p)p^{-2s}) = \prod_{p \nmid q} \frac{(1 - p^{-2s})}{(1 - p^{-s})(1 - \chi_q(p)p^{-s})}.$$

Notice, however, that if $\chi_q(p) = -1$ then this is equal to 1. So we can reduce this to

$$\psi(s) = \prod_{\chi_q(p)=1} \frac{1 + p^{-s}}{1 - p^{-s}} = \prod_{\chi_q(p)=1} (1 + p^{-s}) \sum_{k=0}^{\infty} p^{-ks} = \prod_{\chi_q(p)=1} \left(1 + \sum_{k=1}^{\infty} 2p^{-ks}\right).$$

Applying lemma 15.2, we see that $\psi(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ where $a_n \geq 0$, the series converges for $s > 1$, and $a_1 = 1$.

Now expand ψ as a power series in s about $s = 2$. We know that it is analytic for $\operatorname{Re} s > 1/2$, so the radius of convergence is at least $3/2$. The n-th coefficient will be equal to $\psi^{(n)}(2)/n!$, and

$$\psi^{(n)}(2) = \sum_{m=1}^{\infty} a_m(-\log m)^n m^{-2} = (-1)^n c_n,$$

with $c_n \geq 0$ and $c_0 = \psi(2) = \sum_{n=1}^{\infty} a_n n^{-2} \geq a_1 = 1$. Thus we can write

$$\psi(s) = \sum_{n=1}^{\infty} c_n(2-s)^n \geq c_0 = 1 \quad 1/2 < s < 2.$$

However, this contradicts the fact that $\psi(s) \to 0$ as $s \to 1/2$. So $A(1, \chi_q) \neq 0$. □

5. Conclusion

We have now proven theorem 1. It is possible to prove something stronger: that, on average, there is the same number of primes in the sequences $\{a_1 + kq\}$ and $\{a_2 + kq\}$, assuming that $(a_1, q) = (a_2, q) = 1$. This means that the primes are *equidistributed* modulo q. For an exposition of this fact, as well as a more detailed and algebraic exposition of the proof of this theorem, see [2].

References

[1] Niven, I., Zuckerman, H. S., Montgomery, H. L. *An Introduction to the Theory of Numbers*. New York: John Wiley & Sons, Inc., 5th ed. 1991.

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[3] Ireland, Kenneth. Rosen, Michael. *A Classical Introduction to Modern Number Theory*. New York: Springer-Verlag, 1991. 2nd ed.