HOMEWORK ASSIGNMENT # 9
DUE, Monday, December 15

Collaboration: On the homework sets, collaboration is not only allowed but encouraged. However, you must write up and understand your own individual homework solutions, and you may not share written solutions. If you learn how to solve a problem by talking to a classmate, CA, or looking it up in a book, you should cite the source in your homework write-up, just as you would reference your sources in a literature or history class.
Show all your working, and write up your solutions as neatly as possible.

1. **The p-adic numbers**. Let p be prime. Any non-zero rational number can be written uniquely in the form $r = p^m \cdot a/b$, where $m \in \mathbb{Z}$, a and b are integers coprime to p and to each other with $b > 0$. Define $v(r) = m$, and a "p-adic absolute value" $\|\cdot\|$ as follows:
$$\|r\|_p = \begin{cases} 1/p^m, & r \neq 0, \\ 0, & r = 0. \end{cases}$$
This defines a new notion of size on the rational numbers. Note that under this definition, all integers m satisfy $\|m\|_p \leq 1$, which may seem counterintuitive. The correct way to think of this notion of 'size' is that $\|x - y\|_p$ is small if x and y are congruent modulo some large power of p.

   (a) (2 points) Let x and y be rational numbers. Prove the strong triangle inequality:
$$\|x + y\|_p \leq \max\{\|x\|_p, \|y\|_p\} \leq \|x\|_p + \|y\|_p.$$

   (b) (2 points) Let $x \in \mathbb{Q}$. Say that a sequence $\{x_n\}_{n\in\mathbb{N}}$ *converges p-adically to x* if
$$\lim_{n\to\infty} \|x_n - x\|_p = 0.$$
Prove that any rational number r such that $\|r\|_p \leq 1$ is the limit of a sequence $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \in \mathbb{Z}$.

   (c) (4 points) Say that a sequence of rational numbers $\{x_n\}_{n\in\mathbb{N}}$ is a *p-adic Cauchy sequence* if
$$\lim_{n,m\to\infty} \|x_n - x_m\|_p = 0.$$
Let $x_n \in \mathbb{Z}$ be integers such that $x_n^2 + 1 \equiv 0 \mod 5^n$, and $x_n \equiv x_{n-1} \mod 5^n$. Prove that the sequence $\{x_n\}$ is a 5-adic Cauchy sequence, but that $\{x_n\}$ does not converge to any element of $\mathbb{Q}$.

(d) (4 points) Usually one defines the real numbers $\mathbb{R}$ by taking the set of equivalence classes of all Cauchy sequences with respect to the usual absolute value (in other words, adding to $\mathbb{Q}$ the limit of every Cauchy sequence). Similarly, we may define the rational p-adic numbers $\mathbb{Q}_p$ in the same way. Explicitly, we say that any p-adic Cauchy sequence $\{x_n\}$ defines a p-adic number x, and that two p-adic numbers $x = \{x_n\}$ and $y = \{y_n\}$ are equal if and only if

$$\lim_{n\to\infty} \|x_n - y_n\|_p = 0.$$

Prove that the sequence $\{s_n\}$, with

$$s_n = \sum_{k=1}^{n} a_k$$

converges to a p-adic number if and only if $\lim_{k\to\infty} \|a_k\|_p = 0$. Note this result is not true for $\mathbb{R}$ (the harmonic series), so in some sense convergence in $\mathbb{Q}_p$ is easier than in $\mathbb{R}$.

2. Rational solutions verses integral solutions.

(a) (4 points) Let p and q be odd primes, and let n be a positive integer. Using the Hasse–Minkowski theorem, prove that $x^2 + py^2 = q$ has a solution in rational numbers x, y if and only if $q \equiv 1$ or $p \mod 4$, and

$$\left(\frac{-p}{q}\right) = 1.$$

(b) (4 points) Find p and q such that

$$x^2 + py^2 = q$$

has a solution with $x,\, y \in \mathbb{Q}$, but *not* with $x,\, y \in \mathbb{Z}$. Moreover, for your specific p and q, find all rational solutions explicitly.