MATH 113 – PRACTICE PROBLEMS FOR THE FINAL
MAY, 2005

(1) Let $f(z) = \frac{e^z}{(z-2)(z-4)}$. Find all poles of f and their residues. Compute the integral

$$\int_{|z|=r} f(z)dz$$

for $r = 1, 4$, and 30.

(2) Construct a conformal map from the unit disk onto the first quadrant $\{z \mid 0 < \arg z < \pi/2\}$ such that $f(0) = 2e^{i\pi/4}$.

(3) Let $U = \{z = x + iy \mid |z - 3i| < 5 \text{ and } y > 0\}$. Sketch U. Find a fractional linear transformation f taking U to an angular sector, and such that $f(0) = 1$.

(4) Evaluate the following integral using the method of residue calculus, for $a > 0$. Justify your steps.

$$\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2 + a^2)} dx.$$

(5) Compute the Laurent series for $f(z) = \frac{z+1}{z-1}$ on
   (a) the annulus $\{z \mid 0 < |z| < 1\}$
   (b) the annulus $\{z \mid 1 < |z| < \infty\}$
   Evaluate $\text{Res}(f; 0)$ and $\text{Res}(f; \infty)$.

(6) Suppose that $f = u + iv$ is analytic on the unit disk. Show that

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta})\, d\theta$$

for all $0 < r < 1$.

(7) (a) Is there a conformal map of $\mathbb{C}$ onto the unit disk?
   (b) Is there a conformal map of the unit disk onto $\mathbb{C} - \{0\}$?
   (c) Does there exist a nonconstant bounded analytic function defined on $\mathbb{C} - \mathbb{Z}$?

(8) How many roots does the equation $e^z - 4z^n + 1 = 0$ have in the open unit disk (n is a natural number)?

(9) Suppose that f is analytic on the open disk $|z| < R$ and that $|f(z)| < M$ for $|z| < R$. Suppose also that $f(z_0) = w_0$, where z_0 is some point in the given disk. Show that:

$$\left|\frac{M(f(z) - w_0)}{M^2 - \overline{w}_0 f(z)}\right| \leq \left|\frac{R(z - z_0)}{R^2 - \overline{z}_0 z}\right|.$$