PRACTICE MIDTERM EXAM
Solutions

1. Let $\zeta = e^{\pi i/2n}$ be a primative $4n$th root of unity.

   (a) Prove that
   $$\zeta + \zeta^3 + \zeta^5 + \ldots + \zeta^{2n-1} = \frac{i}{\sin(\pi/2n)}$$

   (b) Let C be the rectangle with vertices 1, $1+i$, $-1+i$ and -1. Evaluate the following integral:
   $$\oint_C \frac{dz}{z^{2n}+1}.$$

   The first sum is a geometric series. We compute that
   $$\zeta(1+\zeta^2+\ldots\zeta^{2n-2}) = \zeta\cdot\left(\frac{\zeta^{2n}-1}{\zeta^2-1}\right) = \frac{-2\zeta}{\zeta^2-1},$$
   where we use the fact that $\zeta^{2n} = e^{\pi i} = -1$. Manipulating this expression we see that:
   $$\frac{-2\zeta}{\zeta^2-1} = \frac{-2}{\zeta-\zeta^{-1}} = \frac{2i\cdot i}{e^{\pi i/2n}-e^{-\pi i/2n}} = \frac{i}{\sin(\pi/2n)}.$$

   The poles of $1/(z^{2n}+1)$ are exactly the zeros of $z^{2n}=-1$, which are given by ζ^{2k-1} for $k=1,2,\ldots 2n$, where $\zeta = e^{\pi i/2n}$. The odd powers of ζ contained inside C are exactly the first n odd powers. Since all the poles are simple poles, the residue at $\omega = \zeta^{2k-1}$ is given by
   $$\lim_{z\to\omega}\frac{(z-\omega)}{z^{2n}+1} = \lim_{z\to\omega}\frac{1}{2nz^{2n-1}} = \frac{\omega}{2n\omega^{2n}} = \frac{\zeta^{2k-1}}{-2n}.$$
   Here we have used l'Hôpital's rule, as well as the fact that $\omega^{2n}+1=0$. Thus by the residue theorem,
   $$\oint_C \frac{dz}{z^{2n}+1} = 2\pi i\sum\text{residues} = 2\pi i\sum_{k=1}^{n}\frac{\zeta^{2k-1}}{-2n} = \frac{\pi}{n\sin(2\pi/n)}.$$

2. Let $f(z)$ be a holomorphic function, and let $u(x,y)$ denote the real part of $f(x+iy)$. Show that
   $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

justifying all steps.

Since f is holomorphic, it is infinitely differentiable, and so in particular, v has continuous second partial derivatives. Thus

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}.$$

Using this, the result is a straightfoward application of the Cauchy–Riemann equations, since

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial^2 v}{\partial x \partial y},$$

and

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) = -\frac{\partial^2 v}{\partial y \partial x}.$$

3. Let $f(z)$ be holomorphic, except for a singularity at $z = 0$.

   (a) Prove that if $1/f(z)$ is bounded in some neighbourhood of zero, then $f(z)$ has a pole of finite order at $z = 0$.

   (b) Suppose that $f(z)$ has an essential singularity at 0. Let $\lambda \in \mathbb{C}$ be any complex number. Show that in any neighbourhood of 0, $f(z) - \lambda$ becomes arbitrarily close to 0.

Suppose that $1/f(z)$ is bounded in a neighbourhood of 0. Then by Riemann's removable singularity theorem $g(z) := 1/f(z)$ extends to a holomorphic function at 0. Clearly $g(z)$ is not identically 0. Thus by considering the Taylor series of $g(z)$ at $z = 0$ we see that

$$g(z) = a_n z^n + a_{n+1} z^{n+1} + \dots$$

for some $a_n \neq 0$. In particular,

$$\lim_{z \to 0} \frac{g(z)}{z^n} = a_n \neq 0.$$

Inverting this limit we see that:

$$\lim_{z \to 0} z^n f(z) = \frac{1}{a_n} \neq 0.$$

Thus $f(z)$ has a pole of order n at 0. (If $n = 0$, then $f(z)$ would not have a singularity at $z = 0$).

If $f(z)$ has a essential singularity at 0 then clearly $f(z) - \lambda$ does also. Assume that $f(z) - \lambda$ does not become arbitrarily close to zero. Then $1/(f(z) - \lambda)$ is bounded in a neighbourhood of 0, and so by part (a), $f(z) - \lambda$ has a pole of finite order at $z = 0$. This is a contradiction, so $f(z) - \lambda$ must become arbitrarily close to zero.

4. Let $f(z)$ be an entire function such that $f(z+1) = f(z)$.

   (a) By taking the branch cut of $\log z$ along the negative real axis, the function

   $$F(z) := f\left(\frac{\log z}{2\pi i}\right)$$

   becomes holomorphic for all $z \in \mathbb{C} \setminus [-\infty, 0]$. Prove that $F(z)$ extends to a holomorphic function for all $z \neq 0$.

   (b) By letting $z = e^{2\pi i \tau}$, show using the Laurent series for $F(z)$ that for all $\tau \in \mathbb{C}$,

   $$f(\tau) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i \tau},$$

   where

   $$a_n = \int_0^1 f(z) e^{-2\pi i n z} dz.$$

We define a second function $G(z)$, as follows. Let

$$G(z) := f\left(\frac{\log' z}{2\pi i}\right)$$

after taking some branch cut of $\log z$ along the *positive* real axis. Then clearly $G(z)$ is holomorphic for all $z \in \mathbb{C} \setminus [0, \infty]$. If for all z in the common domain of F and G we can prove that $F(z) = G(z)$, it follows that $F(z)$ extends to a holomorphic function on the negative real axis by defining $F(-x) = G(-x)$. To compare $F(z)$ and $G(z)$, note that any two different branches of $\log z$ differ by some integer multiple of $2\pi i$. Thus

$$G(z) - F(z) = f\left(\frac{\log' z}{2\pi i}\right) - f\left(\frac{\log z}{2\pi i}\right) = f\left(\frac{\log z + 2\pi i n}{2\pi i}\right) - f\left(\frac{\log z}{2\pi i}\right).$$

Yet for $n \in \mathbb{Z}$, $f(\tau + n) - f(\tau) = 0$, and so

$$G(z) - F(z) = f\left(\frac{\log z}{2\pi i} + n\right) - f\left(\frac{\log z}{2\pi i}\right) = 0.$$

Since $F(z)$ is holomorphic for all $z \neq 0$, It has a Laurent Series which converges for all $z \neq 0$. In particular,

$$F(z) = \sum_{-\infty}^{\infty} a_n z^n, \qquad a_n = \frac{1}{2\pi i} \oint_C \frac{F(z)}{z^{n+1}} dz.$$

for some loop C around 0. Let C be the unit circle in $\mathbb{C}$. Make the substitution $z = e^{2\pi i \tau}$ in the integral for a_n. Then

$$a_n = \frac{1}{2\pi i} \int_0^1 \frac{2\pi i \cdot e^{2\pi i \tau} F(e^{2\pi i \tau})}{e^{2\pi i (n+1)\tau}} d\tau = \int_0^1 F(e^{2\pi i \tau}) e^{-2\pi i n \tau} d\tau.$$

On the other hand,

$$F(e^{2\pi i \tau}) = f\left(\frac{\log(e^{2\pi i \tau})}{2\pi i}\right) = f(\tau).$$

Thus we are done.

5. Let $P(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ be a polynomial of degree n. Let R be a region containing all the zeros of $P(x)$, and let C be the boundary of R. Evaluate the integral

$$\oint_C \frac{xP'(x)}{P(x)} dx.$$

Since the coefficient of x^n in $P(x)$ is 1, we may write

$$P(x) = \prod_{i=1}^{n} (x - \alpha_i),$$

where α_i are the roots of $P(x)$ (with multiplicity). As in class, we find that

$$\frac{P'(x)}{P(x)} = \sum_{i=1}^{n} \frac{1}{z - \alpha_i}.$$

Thus since all the roots lie within the contour, from the residue theorem we see that

$$\oint \frac{xP'(x)}{P(x)} = \oint \sum_{i=1}^{n} \frac{x}{x - \alpha_i} dx = 2\pi i \sum_{i=1}^{n} \alpha_i.$$

On the other hand, we have that

$$P(x) = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n) = x^n - (\alpha_1 + \alpha_2 + \ldots + \alpha_n)x^{n-1} + \ldots$$

and so

$$\oint \frac{xP'(x)}{P(x)} = -2\pi i a_{n-1}.$$