MIDTERM EXAM
Solutions

1. Let f be a holomorphic function, and let $u(x,y)$ be the real part of $f(x+iy)$. Suppose that $u(x,y)=u(-y,x)$. Prove that for all $z\in\mathbb{C}$, $f(z)=f(iz)$.

Let $g(z)=f(z)-f(iz)=f(x+iy)-f(-y+ix)$. Write $f=u_f+iv_f$ and $g=u_g+iv_g$. Then $u_g(x,y)=u_f(x,y)-u_f(-y,x)=0$. Thus $u_g=0$, and by the Cauchy–Riemann equations:

$$\frac{\partial v_g}{\partial x}=0, \qquad \frac{\partial v_g}{\partial y}=0.$$

Hence v_g is constant. Thus $g(z)$ is constant. Letting $z=0$, we see that $g(z)=f(0)-f(0)=0$. Thus $g(z)=0$, and $f(z)=f(iz)$.

2. Let $f(z)$ be a holomorphic function such that $f(z)$ is real for $z\in\mathbb{R}$. Prove that for all $z\in\mathbb{C}$,

$$f(\overline{z})=\overline{f(z)}.$$

Since $f(z)$ is real, all the derivatives of $f(z)$ at 0 are also real. Thus taking the Taylor series expansion around $z=0$, we find that

$$f(z)=\sum_{n=0}^{\infty}a_nz^n$$

with $a_n\in\mathbb{R}$. Since $\overline{a_n}=a_n$, we explicitly see from this formula that $f(\overline{z})=\overline{f(z)}$.

3. Let C denote the unit circle in $\mathbb{C}$, and let $g(z)$ be a bounded integrable (but not necessarily holomorphic!) function on C. For all a inside C, let:

$$g(a)=\frac{1}{2\pi i}\oint\frac{f(z)}{z-a}dz.$$

(a) Prove that $g(a)$ is holomorphic for $|a|<1$.

(b) Let $f(z)=1$ for $\mathrm{Im}\{z\}\geq 0$ and 0 for $\mathrm{Im}\{z\}<0$. Explicitly compute $g(a)$.

We want to show that $g(a)$ is differentiable as a function of a complex variable. Thus we write

$$\frac{g(a+h)-g(a)}{h}=\frac{1}{2\pi ih}\left(\oint\frac{f(z)}{z-a-h}dz-\oint\frac{f(z)}{z-a}dz\right)$$

which simplifies to

$$\frac{g(a+h)-g(a)}{h} = \frac{1}{2\pi i}\oint \frac{f(z)}{(z-a)(z-a-h)}dz$$

To show g is differentiable at a it suffices to show that the limit as h approaches 0 exists. We have done this in class several times, but to repeat the argument here, write

$$\oint \frac{f(z)}{(z-a)(z-a-h)} = \oint \frac{f(z)}{(z-a)^2} + h\oint \frac{f(z)}{(z-a)^2(z-a-h)}.$$

The second integral is bounded by the ML inequality. Thus as h approaches zero, the second term vanishes, and we are done.

For part (b), we see that

$$g(a) = \frac{1}{2\pi i}\int_1^{-1} \frac{1}{z-a}dz = \left[\frac{\log(z-a)}{2\pi i}\right]_1^{-1} = \frac{1}{2\pi i}\left(\log(-1-a) - \log(1-a)\right).$$

We have to choose a branch of log that is holomorphic in the upper semicircle, but any such choice of log gives the same answer, since extra factors of $2\pi i$ in one term are subtracted in the other. One can also write this answer as

$$g(a) = \frac{1}{2\pi i}\left(\pi i + \log\left(\frac{1+a}{1-a}\right)\right) = \frac{1}{2} + \frac{1}{2\pi i}\log\left(\frac{1+a}{1-a}\right)$$

where the branch of log is 0 at 1 (i.e. $g(0) = 1/2$).

4. Let $P(z)$ be a polynomial of degree at least two. Let C bound a disc such that all the zeros of $P(z)$ lie within C. Prove that

$$\oint_C \frac{dz}{P(z)} = 0.$$

Since all the zeros of $P(z)$ lie within C, the function $1/P(z)$ has no poles outside C. Thus we may increase the radius of C without affecting the value of the integral. Let $P(z)$ be of degree d. If $|z| = R$, then for sufficiently large R, $|P(z)| > cR^d$ for some constant c. Thus by the ML-inequality,

$$\left|\oint_C \frac{dz}{P(z)}\right| \leq \frac{2\pi R}{cR^d} = \frac{2\pi}{cR^{d-1}}.$$

In particular, if $d \geq 2$, then the absolute value of this integral is less than any positive number, and thus must equal zero. Hence

$$\oint_C \frac{dz}{P(z)} = 0.$$

5. (a) What are the complex zeros of $\cos z - \sin z$?

(b) What is the radius of convergence of the Taylor series of

$$f(z) = \frac{1}{\cos z - \sin z}$$

around $z = 0$?

Let $t = e^{iz}$. Then $\cos z = \frac{t+t^{-1}}{2}$ and $\sin z = \frac{t-t^{-1}}{2i}$. Solving the equation

$$\frac{t+t^{-1}}{2} = \frac{t-t^{-1}}{2i}$$

we find that $t^2 = i$, or $e^{2iz} = e^{\pi i/2}$. Taking the log of both sides, we find that

$$2iz = \pi i/2 + 2\pi i k,$$

or in other words, $z = \pi/4 + k\pi$, for integral k. Note that these zeros are all real!

The radius of convergence of $f(z)$ is the distance from zero of the nearest singularity. This occurs at the smallest zero of $\cos z - \sin z$, which by part (a) is $\pi/4$.