Problem Set 8 Solution Set

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Math 113: Complex Analysis, Fall 2002

1. Prove that
$$\sum_{n=2}^{\infty} \frac{1}{\binom{n}{2}^3} = \frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{6^3} + \frac{1}{10^3} + \cdots = 8(10 - \pi^2).$$

Solution I: The "easy way" for contest math lovers. First note that
$$\sum_{n=2}^{\infty} \frac{1}{\binom{n}{2}^3} = \sum_{n=2}^{\infty} \left(\frac{2}{n(n-1)} \right)^3 = 8 \sum_{n=2}^{\infty} \frac{1}{n^3(n-1)^3}.$$
Now we use partial fractions to decompose the fraction $1/n^3(n-1)^3$ as follows:
$$\frac{1}{n^3(n-1)^3} \equiv \frac{A}{n} + \frac{B}{n^2} + \frac{C}{n^3} + \frac{D}{(n-1)} + \frac{E}{(n-1)^2} + \frac{F}{(n-1)^3},$$
$$1 \equiv n^2(n-1)^3 A + n(n-1)^3 B + (n-1)^3 C + n^3(n-1)^2 D + n^3(n-1)E + n^3 F.$$
Since both sides are identical, we may substitute certain convenient values of n and solve for the coefficients $A, \ldots, F$. For example, putting $n = 0$ we obtain $C = -1$. In this way we compute
$$A = -6, \quad B = -3, \quad C = -1, \quad D = 6, \quad E = -3, \quad F = 1.$$
Hence
$$\begin{aligned}
8 \sum_{n=2}^{\infty} \frac{1}{n^3(n-1)^3} &= 8 \sum_{n=2}^{\infty} \left(\frac{-6}{n} + \frac{-3}{n^2} + \frac{-1}{n^3} + \frac{6}{(n-1)} + \frac{-3}{(n-1)^2} + \frac{1}{(n-1)^3} \right), \\
&= 8 \sum_{n=2}^{\infty} \left(\frac{-6}{n} + \frac{-3}{n^2} + \frac{-1}{n^3} \right) + 8 \sum_{n=1}^{\infty} \left(\frac{6}{n} + \frac{-3}{n^2} + \frac{1}{n^3} \right) \\
&= 8 \left(6 + 3 + 1 - 6 \sum_{n=1}^{\infty} \frac{1}{n^2} \right) \\
&= 8(10 - 6(\pi^2/6)) = 8(10 - \pi^2).
\end{aligned}$$

□

Solution II: Using complex analysis (Benjamin Bakker). Let $f(z) = 8/z^3(z-1)^3$. We want to show $\sum_{n=2}^{\infty} f(n) = 8(10 - \pi^2)$ Since $f(-n) = f(n-1)$ we have
$$\sum_{n \neq 0,1} f(n) = 2 \sum_{n=2}^{\infty} f(n).$$

Recall we showed in class that

$$\sum{}' f(n) = -\sum \text{ residues of } \pi f(z)\cot \pi z \text{ at poles of } f(z),$$

where the sum $\sum'$ is taken over poles of $\cot \pi z$ which are now poles of f. In our case then,

$$\sum_{n=2}^{\infty} \frac{1}{\binom{n}{2}^3} = \frac{1}{2} \sum_{n\neq 0,1} f(n) = -\frac{1}{2}\left(\mathrm{Res}_{z=0}\ \pi f(z)\cot \pi z + \mathrm{Res}_{z=1}\ \pi f(z)\cot \pi z\right).$$

Let's compute the residue at $z = 0$. We can assume that $|z| \leq 1$ in this region and use a power series expansion for f:

$$f(z) = -\frac{8}{z^3}\left(\frac{1}{z-1}\right)^3 = -\frac{8}{z^3}(1 + 3z + 6z^2 + 10z^3 + \cdots).$$

On the other hand,

$$\cot \pi z = \frac{1}{\pi z}\left(1 - \frac{\pi^2 z^2}{3} + \cdots\right).$$

Recall that the residue of a power series expansion is the coefficient of z^{-1}, so that

$$\begin{aligned}\mathrm{Res}_{z=0}\ \pi f(z)\cot \pi z &= z^{-1} \text{ coeff of } -\frac{8\pi}{\pi z^4}(1 + 3z + 6z^2 + 10z^3 + \cdots)\left(1 - \frac{\pi^2 z^2}{3} + \cdots\right) \\ &= -8(10 - \pi^2).\end{aligned}$$

Now we compute the residue at $z = 1$. We can easily check that $\cot \pi(z-1) = \cot \pi z$ from which we deduce the expansion

$$\cot \pi(z-1) = \frac{1}{\pi(z-1)}\left(1 - \frac{\pi^2(z-1)^2}{3} + \cdots\right).$$

Around $z = 1$ the quantity $z - 1$ has a small magnitude so we may expand f as follows:

$$f(z) = \frac{8}{(z-1)^3}\left(\frac{1}{(z-1)+1}\right)^3 = \frac{8}{(z-1)^3}(1 - 3(z-1) + 6(z-1)^2 - 10(z-1)^3 + \cdots).$$

Therefore

$$\begin{aligned}\mathrm{Res}_{z=1}\ \pi f(z)\cot \pi z &= (z-1)^{-1} \text{ coeff of} \\ &-\frac{8\pi}{\pi(z-1)^4}(1 - 3(z-1) + 6(z-1)^2 - 10(z-1)^3 + \cdots)\left(1 - \frac{\pi^2(z-1)^2}{3} + \cdots\right) \\ &= -8(10 - \pi^2).\end{aligned}$$

Finally,

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{1}{\binom{n}{2}^3} &= -\frac{1}{2}\left(\mathrm{Res}_{z=0}\ \pi f(z)\cot \pi z + \mathrm{Res}_{z=1}\ \pi f(z)\cot \pi z\right), \\ &= -\frac{1}{2}\left(-8(10 - \pi^2) - 8(10 - \pi^2)\right) \\ &= 8(10 - \pi^2).\end{aligned}$$

$\square$

Figure 1: The contour of integration for problem 3.

By Cauchy's integral formula,

$$\lim_{R\to\infty} \oint_\Gamma \frac{e^{iaz}}{e^z+1}\,dz = 2\pi i \sum_{n=0}^{\infty} \left(\operatorname*{Res}_{z=(2n+1)\pi i} \frac{e^{iaz}}{e^z+1} \right)$$

Let's calculate an individual residue:

$$\begin{aligned}
\operatorname*{Res}_{z=(2n+1)\pi i} \frac{e^{iaz}}{e^z+1} &= \lim_{z\to(2n+1)\pi i} \frac{e^{iaz}(z-(2n+1)\pi i)}{e^z+1} \\
&= \lim_{z\to(2n+1)\pi i} \frac{e^{iaz}+iae^{iaz}(z-(2n+1)\pi i)}{e^z} \\
&= -e^{-a(2n+1)\pi}.
\end{aligned}$$

Substituting this expression above we get

$$\lim_{R\to\infty} \oint_\Gamma \frac{e^{iaz}}{e^z+1}\,dz = 2\pi i \sum_{n=0}^{\infty} -e^{-a(2n+1)\pi} = 2\pi i \frac{e^{-\pi a}}{1-e^{-2\pi a}} = \frac{\pi i}{\sinh \pi a}.$$

Now we separate the contour integral into its component pieces:

$$\begin{aligned}
\oint_\Gamma \frac{e^{iaz}}{e^z+1}\,dz &= \int_0^R \frac{e^{iaz}}{e^z+1}\,dz + \int_{\Gamma_R} \frac{e^{iaz}}{e^z+1}\,dz + \text{ c.p.v.} \int_{Ri}^0 \frac{e^{iaz}}{e^z+1}\,dz + \int_{\text{bumps}} \\
&= \int_0^R \frac{e^{iaz}}{e^z+1}\,dz + \int_0^{\pi/2} \frac{e^{iaRe^{i\theta}}}{e^{Re^{i\theta}}} ie^{i\theta}\,dz + \text{ c.p.v. } - i\int_R^0 \frac{e^{-az}}{e^{iz}+1}\,dz + \int_{\text{bumps}}.
\end{aligned}$$

Let us compute an integral around a typical 'bump' as $\epsilon \to 0$:

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{\pi/2}^{3\pi/2} \frac{e^{ia((2n+1)\pi i+\epsilon e^{i\theta})} i\epsilon e^{i\theta}}{e^{((2n+1)\pi i+\epsilon e^{i\theta})}}\,d\theta &= \int_{\pi/2}^{3\pi/2} \lim_{\epsilon\to 0} \frac{e^{ia((2n+1)\pi i+\epsilon e^{i\theta})} i\epsilon e^{i\theta}}{e^{((2n+1)\pi i+\epsilon e^{i\theta})}}\,d\theta \\ &= \int_{\pi/2}^{3\pi/2} \frac{ie^{-a(2n+1)\pi}}{-1}\,d\theta = -\pi i e^{-a(2n+1)\pi}.\end{aligned}$$

Note that this is exactly half the contribution of a pole for the entire contour integral (see the remark below). An easy computation with the ML inequality show that the integral over Γ_R is zero. Hence

$$\begin{aligned}\lim_{R\to\infty,\epsilon\to 0}\oint_\Gamma \frac{e^{iaz}}{e^z+1}\,dz &= \int_0^\infty \frac{e^{iaz}}{e^z+1}\,dz + \text{ c.p.v. } -i\int_\infty^0 \frac{e^{-az}}{e^{iz}+1}\,dz - \pi i\sum_{n=0}^\infty e^{-a(2n+1)\pi} \\ -2\pi i\sum_{n=0}^\infty e^{-a(2n+1)\pi} &= \int_0^\infty \frac{e^{iaz}}{e^z+1}\,dz + \text{ c.p.v. } -i\int_\infty^0 \frac{e^{-az}}{e^{iz}+1}\,dz - \pi i\sum_{n=0}^\infty e^{-a(2n+1)\pi} \\ -\frac{\pi i}{2\sinh \pi a} &= \int_0^\infty \frac{e^{iaz}}{e^z+1}\,dz + \text{ c.p.v. } -i\int_\infty^0 \frac{e^{-az}}{e^{iz}+1}\,dz.\end{aligned}$$

The idea is to take imaginary parts of both sides above. First, we compute the imaginary part of the integral

$$\text{c.p.v. } -i\int_\infty^0 \frac{e^{-az}}{e^{iz}+1}\,dz.$$

To it:

$$\begin{aligned}-i\int_\infty^0 \frac{e^{-az}}{e^{iz}+1}\,dz &= -i\int_\infty^0 \frac{e^{-az}}{e^{iz}+1}\frac{e^{-iz}+1}{e^{-iz}+1}\,dz \\ &= -i\int_\infty^0 \frac{e^{-az}(e^{-iz}+1)}{2+e^{iz}+e^{-iz}} \\ &= -i\int_\infty^0 \frac{e^{-az}(1+\cos\theta - i\sin\theta)}{2+2\cos\theta}\,dz.\end{aligned}$$

Taking imaginary parts we see that

$$\text{Im c.p.v. } -i\int_\infty^0 \frac{e^{-az}}{e^{iz}+1}\,dz = -\int_\infty^0 \frac{e^{-az}}{2} = -\frac{1}{2a}.$$

Finally,

$$-\frac{\pi}{2\sinh \pi a} = \int_0^\infty \frac{\sin iaz}{e^z+1}\,dz - \frac{1}{2a},$$

which is the desired result. □

Remark. An important moral: Integrating in a straight line *through* a pole contributes exactly half the residue. This is a general useful principle, and could have been applied to other contours, such as a rectangle in the first quadrant with only one bump, as many of you did.

4. Prove that

$$\int_0^1 \log\Gamma(z)\,dz = \frac{1}{2}\log(2\pi)$$

Solution. Recall the useful identity

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z}.$$

From it we deduce that

$$\begin{aligned}
\int_0^1 \log\Gamma(z)\,dz &= \int_0^1 \log\left(\frac{\pi}{\sin\pi z}\frac{1}{\Gamma(1-z)}\right)\,dz \\
&= \int_0^1 \log\pi - \log\sin\pi z - \log\Gamma(1-z)\,dz \\
&= \log\pi - \frac{1}{\pi}\int_0^\pi \log\sin\theta\,d\theta - \int_1^0 \log\Gamma(u){-du} \\
&= \log\pi - \frac{2}{\pi}\int_0^{\pi/2} \log\sin\theta\,d\theta - \int_0^1 \log\Gamma(u)\,du.
\end{aligned}$$

And so

$$\int_0^1 \log\Gamma(z)\,dz = \frac{1}{2}\left(\log\pi - \frac{2}{\pi}\int_0^{\pi/2}\log\sin\theta\,d\theta\right).$$

In the text, we find on p.188 that

$$\int_0^{\pi/2} \log\sin x\,dx = -\frac{\pi}{2}\log 2.$$

From this equality we easily get the result:

$$\begin{aligned}
\int_0^1 \log\Gamma(z)\,dz &= \frac{1}{2}\left(\log\pi + \frac{2}{\pi}\frac{\pi}{2}\log 2\right) \\
&= \frac{1}{2}(\log\pi + \log 2) \\
&= \frac{1}{2}\log 2\pi.
\end{aligned}$$

We briefly sketch how to compute the integral we borrowed from the text. Consider the integral

$$\oint_\Gamma \frac{\log(z+i)}{z^2+1}\,dz$$

where Γ is the usual semicircular path of radius R in the upper half plane centered about the origin. The only pole inside the region bounded by Γ is $z = i$, and the residue there is $(1/2i)\log 2i$. An ML calculation shows that $\int_{\Gamma_R} = 0$. Hence

$$\left(\int_{-\infty}^0 + \int_0^\infty\right)\frac{\log(z+i)}{z^2+1}\,dz = 2\pi i\left(\frac{1}{2i}\log 2i\right) = \pi\log 2i.$$

Substitute $z \to -z$ in the first integral and take real parts to obtain.

$$\int_0^\infty \frac{\log(z^2+1)}{z^2+1}\,dz = \pi \log 2.$$

The substitution $z = \cot \pi x$ yields

$$\int_0^{1/2} \log \sin \pi x \, dx = -\frac{1}{2} \log 2.$$

A linear re-scaling gives the desired integral. $\square$