Problem Set 7 Solution Set

Anthony Varilly

Math 113: Complex Analysis, Fall 2002

1. Let $P(z)$ be a polynomial. Prove there exists a real positive number ϵ with the following property: for all non-zero complex numbers $|\lambda| < \epsilon$, the polynomial $P(z) + \lambda$ has distinct roots.

   Remark. In characterisic zero, a polynomial $Q(z)$ has a repeated root at z_0 if and only if $Q(z_0) = Q'(z_0) = 0$. This is especially easy to verify in $\mathbb{C}$ since we can factor out Q into linear terms (because $\mathbb{C}$ is algebraically closed) and then apply the product rule of differentiation to Q.

   If the polynomial is constant, the result is trivial. In the sequel, we assume P is of degree at least 1.

   Solution 1 (Due to Dustin Cartwright). Suppose $P(z) + \lambda$ has a double root at z_0. Then by the above remark we must have $P(z_0) + \lambda = P'(z_0) = 0$. Hence $\lambda \in \{-P(a_1), \ldots, -P(a_n)\}$, where the a_i are the roots of $P'(z)$, i.e., λ can take on finitely many different values.

   Let ϵ be the smallest distance from the origin to the non-zero points in the set

   $$\{-P(a_1), \ldots, -P(a_n)\}.$$

   Then for non-zero $|\lambda| < \epsilon$ the polynomial $P(z) + \lambda$ can't have multiple roots. Note that it's OK if some of the $-P(a_i)$'s are zero since we want non-zero λ (if all of them are zero then ϵ can be arbitrary). □

   Solution 2 (Due to John Provine and Benjamin Bakker). Let n be the degree of $P(z)$. We claim that for each root α of P there is a neighborhood N_α of α such that P has non-zero derivative at all points of N_α except possibly α. Otherwise α would be an accumulation point in $\mathbb{C}$ of the set of roots of P, but P has finitely many roots, so there are no accumulation points of this set. Moreover, we may suppose the boundary ∂N_α contains no roots of P.

   Since ∂N_α is compact and $|P|$ is continuous, the polynomial achieves a minimum on this boundary, which we will denote ϵ_α. Let $\epsilon = \min_\alpha\{\epsilon_\alpha\}$. Then for non-zero $|\lambda| < \epsilon$ we have

   $$|P(z)| \geq \epsilon_\alpha \geq \epsilon > |\lambda| \quad \text{for all } z \in \partial N_\alpha.$$

   Since the constant function $z \mapsto \lambda$, Rouche's Theorem tells us P and $P + \lambda$ have the same number of roots inside ∂N_α, i.e., in N_α. Since $P(\alpha) + \lambda = \lambda \neq 0$, we must have the roots of $P + \lambda$ inside the open sets $N_\alpha - \{\alpha\}$. However, by construction, the derivative $(P + \lambda)' = P'$ is non-zero in all the open sets $N_\alpha - \{\alpha\}$. So $(P + \lambda)'$ is non-zero at all the roots of $P + \lambda$, which is to say that $P + \lambda$ has distinct roots. □

Solution 3 (Due to Mark Lezama, Tony Varilly and Philip Zeyliger). We use resultants. Let $P_1(z)$ be a polynomial of degree n_1 with roots $\alpha_1, \ldots, \alpha_{n_1}$ and similarly let $P_2(z)$ be a polynomial of degree n_2 with roots $\beta_1, \ldots, \beta_{n_2}$. The *resultant* $R(P_1, P_2)$ of two polynomials is defined as

$$R(P_1, P_2) := \prod_{i=1}^{n_1} \prod_{j=1}^{n_2} (\beta_j - \alpha_i).$$

By the above remark a polynomial P has repeated roots if and only if $R(P, P') = 0$.

Set $Q(z) = P(z) + \lambda$. Let $\alpha_1(\lambda), \ldots, \alpha_n(\lambda)$ be the roots of Q and ;et $\beta_1, \ldots, \beta_{n-1}$ be the roots of $Q' = P'$. Then

$$R(Q, Q') = R(Q, P') = \prod_{i=1}^{n} \prod_{j=1}^{n-1} (\beta_j - \alpha_i(\lambda)).$$

The key here is to notice that $R(Q, Q')$ is a function on the coefficients of Q and Q' (why?), and thus can be regarded as a polynomial in λ. The equation $R(Q, Q') = 0$ has only finitely many solutions and thus there are only finitely many λ such that Q has a double root. As in the first solution, we let $\epsilon = \min\{|\lambda| : R(Q, Q') = 0, \lambda \neq 0\}$, and this ϵ has our desired property. $\square$

2. Prove that if m is a real number such that $m > 0$, then

$$\int_0^\infty \frac{\cos mx}{(x^2+1)^2}\, dx = \frac{\pi e^{-m}(1+m)}{4}.$$

Solution. It is enough to prove that

$$\int_{-\infty}^\infty \frac{\cos mx}{(x^2+1)^2}\, dx = \frac{\pi e^{-m}(1+m)}{2}$$

because the integrand is an even function. Consider the integral

$$\oint_C \frac{e^{imz}}{(z^2+1)^2}\, dz,$$

where C is the usual semicircular contour of radius R in the upper half plane. Since

$$\lim_{|z|\to\infty} \left| \frac{1}{(z^2+1)^2} \right| = 0,$$

applying Jordan's lemma we see that

$$\int_{-\infty}^\infty \frac{e^{imx}}{(x^2+1)^2}\, dx = 2\pi i \sum \text{residues of } \frac{e^{imz}}{(z^2+1)^2} \text{ in the upper half-plane}$$

Now, the only pole of the integrand is at $z = i$; it is a double pole. The residue at i is

$$\begin{aligned}
\lim_{z\to i} \frac{d}{dz} \frac{e^{imz}(z-i)^2}{(z^2+1)^2} &= \lim_{z\to i} \frac{d}{dz} \frac{e^{imz}}{(z+i)^2} \\
&= \lim_{z\to i} \frac{ime^{imz}(z+i)^2 - 2e^{imz}(z+i)}{(z+i)^4} \\
&= \frac{e^{-m}(m+1)}{4i}.
\end{aligned}$$

Hence

$$\int_{-\infty}^{\infty} \frac{\cos mx}{(x^2+1)^2}\,dx = \mathrm{Re}\left(\int_{-\infty}^{\infty} \frac{e^{imx}}{(x^2+1)^2}\,dx\right) = \mathrm{Re}\left(2\pi i \cdot \frac{e^{-m}(m+1)}{4i}\right) = \frac{\pi e^{-m}(m+1)}{2}.$$

$\square$

3. If $a^2 > b^2 + c^2$, prove that

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta + c\sin\theta} = \frac{2\pi}{\sqrt{a^2 - b^2 - c^2}}.$$

Solution. We first make the substitution $z = e^{i\theta}$, so that $\cos\theta = (z + 1/z)/2$ and $\sin\theta = (z - 1/z)/2i$:

$$\begin{aligned}\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta + c\sin\theta} &= -2i \oint_{|z|=1} \frac{dz/z}{2a + b(z+1/z) - ci(z - 1/z)} \\ &= \oint_{|z|=1} \frac{2dz}{(bi+c)z^2 + 2aiz + (bi - c)}.\end{aligned}$$

To find the poles of the above integrand inside the unit circle we use the quadratic formula:

$$z_\pm = \frac{-a \pm \sqrt{a^2 - b^2 - c^2}}{b - ci}.$$

Now notice that $|z_+|\,|z_-| = |(bi - c)/(bi + c)| = 1$. Hence, either $|z_+| = |z_-| = 1$ or one of z_+, z_- is inside the unit circle and the other one is outside. Say, for example, that $|z_+| = 1$. Then

$$\begin{aligned}|-a + \sqrt{a^2 - b^2 - c^2}| &= |b - ci| \\ a - \sqrt{a^2 - b^2 - c^2} &= \sqrt{b^2 + c^2} \\ 2a^2 - 2\sqrt{a^2 - b^2 - c^2} &= 2b^2 + 2c^2\end{aligned}$$

Hence

$$a^2 - b^2 - c^2 = \sqrt{a^2 - b^2 - c^2}, \tag{1}$$

Since $|z_-|$ must also be 1 in this case, a similar computation shows that

$$-a^2 + b^2 + c^2 = \sqrt{a^2 - b^2 - c^2}. \tag{2}$$

The only way to reconcile (1) and (2) is if $a^2 = b^2 + c^2$ but $a > b^2 + c^2$ and this is a contradiction. A quick calculation shows that $|z_-| > |z_+|$, so $|z_+|$ is inside the unit circle. We need only calculate the residue at z_+ to compute the desired integral:

$$\begin{aligned}\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta + c\sin\theta} &= 2\pi i \cdot \lim_{z\to z_+} \frac{2(z - z_+)}{(bi + c)z^2 + 2aiz + (bi - c)} \\ &= \lim_{z\to z_+} \frac{4\pi i}{2(bi + c)z + 2ai} \\ &= \frac{2\pi i}{(-a + \sqrt{a^2 - b^2 - c^2})i + ai} \\ &= \frac{2\pi}{\sqrt{a^2 - b^2 - c^2}}.\end{aligned}$$

□

4. Let $\lambda = a + bi$ with $a > 0$.

   (a) Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x\sin x}{x^2 + \lambda^2}\, dx.$$

Solution. First, we exapnd the integral as

$$\int_{-\infty}^{\infty} \frac{x\sin x}{x^2 + \lambda^2}\, dx = \frac{1}{2i}\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2 + \lambda^2}\, dx - \frac{1}{2i}\int_{-\infty}^{\infty} \frac{xe^{-ix}}{x^2 + \lambda^2}\, dx.$$

For the first integral in the right hand side, note that

$$\lim_{|z|\to\infty} \left|\frac{z}{z^2 + \lambda^2}\right| = 0,$$

so we may apply Jordan's lemma to conclude that

$$\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2 + \lambda^2}\, dx = 2\pi i \sum \text{residues of } \frac{ze^{iz}}{z^2 + \lambda^2}.$$

The only residue in the upper half plane is $z = \lambda i = -b + ai$ because $a > 0$. Hence

$$\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2 + \lambda^2}\, dx = 2\pi i \cdot \lim_{z\to\lambda i} \frac{ze^{iz}(z - \lambda i)}{z^2 + \lambda^2} = \pi i e^{-\lambda}.$$

Instead of going through a similar computation, we cleverly notice that the change of variables $y = -x$ yields

$$\int_{-\infty}^{\infty} \frac{xe^{-ix}}{x^2 + \lambda^2}\, dx = -\int_{-\infty}^{\infty} \frac{ye^{iy}}{y^2 + \lambda^2}\, dy = -\pi i e^{-\lambda}.$$

We conclude that

$$\int_{-\infty}^{\infty} \frac{x\sin x}{x^2 + \lambda^2}\, dx = \frac{1}{2i}(\pi i e^{-\lambda} + \pi i e^{-\lambda}) = \pi e^{-\lambda}.$$

□

(b) Using your answer to part (a), prove that

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^4 + 4a^4}\, dx = \frac{\pi e^{-a} \sin a}{2a^2}.$$

Solution. We may assume without loss of generality that $a > 0$; the case $a < 0$ is done analogously. (The case $a = 0$ is clear.) We use "pseuso partial fractions":

$$\frac{1}{x^2 + \lambda^2} = \frac{1}{4a^2 i}\left(\frac{1}{x^2 - 2a^2 i} - \frac{1}{x^2 + 2a^2 i}\right).$$

We seek λ_1 and λ_2 with positive real part and such that $\lambda_1^2 = -2a^2 i$ and $\lambda_2^2 = 2a^2 i$. A quick computation shows that $\lambda_1 = a - ai$ and $\lambda_2 = a + ai$. Using part (a) we compute

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{x \sin x}{x^4 + 4a^4}\, dx &= \frac{1}{4a^2 i}\left(\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + \lambda_1^2} - \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + \lambda_2^2}\right) \\
&= \frac{1}{4a^2 i}(\pi e^{-\lambda_1} + \pi e^{-\lambda_2}) \\
&= \frac{1}{4a^2 i}(\pi e^{-a}(\cos a + i \sin a) + \pi e^{-a}(\cos a - i \sin a)) \\
&= \frac{\pi e^{-a} \sin a}{2a^2}.
\end{aligned}$$

□

5. Let p and q be integers with $q > p > 0$.

(a) Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{t^{2p}}{t^{2q} + 1}\, dt.$$

Solution. Consider the integral around the usual semicircular contour C in the upper half-plane

$$\oint_C \frac{z^{2p}}{z^{2q} + 1}\, dz = \int_{-R}^{R} \frac{z^{2p}}{z^{2q} + 1}\, dz + \int_{\Gamma(R)} \frac{z^{2p}}{z^{2q} + 1}\, dz.$$

Since $\deg(z^{2q} + 1) - \deg(t^{2p}) = 2(q - p) \geq 2$, we have

$$\lim_{R \to \infty} \int_{\Gamma(R)} \frac{z^{2p}}{z^{2q} + 1}\, dz = 0.$$

In order to compute integral around C we will use the following fact:

$$\sum_{k=0}^{q-1} e^{\pi i(2k+1)/2q} = \frac{i}{\sin \pi/2q}$$

See problem 2 in Solution Set 6. The function we are integrating has simple poles at $e^{\pi i(2k+1)/2q}$, $k = 0, \dots, 2q - 1$. Only the first q of these poles are inside the region bounded by C. The residue at any one of these poles is

$$\begin{aligned}
\lim_{z\to e^{\pi i(2k+1)/2q}} \frac{z^{2p}(z-e^{\pi i(2k+1)/2n})}{z^{2q}+1} &= \lim_{z\to e^{\pi i(2k+1)/2q}} \frac{z^{2p}}{2qz^{2q-1}} \\
&= \frac{e^{\pi i(2k+1)p/q}}{2qe^{\pi i(2k+1)(2q-1)/2q}} = -\frac{e^{\pi i(2k+1)(2p-2q+1)/2q}}{2q}.
\end{aligned}$$

Hence the sum of the residues of $z^{2p}/(z^{2q}+1)$ inside C is

$$\begin{aligned}
\sum_{k=0}^{q-1} -\frac{e^{\pi i(2k+1)(2p-2q+1)/2q}}{2q} &= -\frac{1}{2q}\sum_{k=0}^{q-1} e^{\pi i(2k+1)[2(p-q)+1]/2q} \\
&= \frac{1}{2q}\sum_{k=0}^{q-1} e^{\pi i(2k+1)(2p+1)/2q} \\
&= \frac{1}{2q}\frac{e^{\pi i(2p+1)/2q}(1--1)}{1-e^{\pi i(2p+1)/q}} \\
&= \frac{1}{2qi\sin(\pi(2p+1)/2q)}
\end{aligned}$$

Therfore

$$\int_{-\infty}^{\infty} \frac{t^{2p}}{t^{2q}+1}\,dt = \lim_{R\to\infty}\int_{-R}^{R} \frac{z^{2p}}{z^{2q}+1}\,dz = \frac{2\pi i}{2qi\sin(\pi(2p+1)/2q)} = \frac{\pi}{q\sin(\pi(2p+1)/2q)}.$$

□

(b) Use part (a) to evaluate the integral

$$\int_0^{\infty} \frac{x^(s-1)}{1+x}\,dx,$$

when $0 < s = (2p+1)/2q < 1$.

Solution. Make the substitution $x = t^{2q}$:

$$\int_0^{\infty} \frac{x^(s-1)}{1+x}\,dx = 2q\int_0^{\infty} \frac{t^{2p(s-1)}}{1+t^{2q}}t^{2q-1}\,dt = 2q\int_0^{\infty} \frac{t^{2p}}{t^{2q}+1}\,dt$$

Using part (a)

$$\int_0^{\infty} \frac{x^(s-1)}{1+x}\,dx = q\int_{-\infty}^{\infty} \frac{t^{2p}}{t^{2q}+1}\,dt = q\,\frac{\pi}{q\sin(\pi(2p+1)/2q)} = \frac{\pi}{\sin(\pi s)}.$$

□