# Problem Set 3 Solution Set

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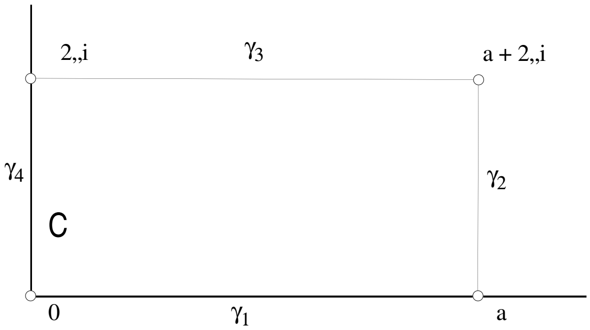
*Math 113: Complex Analysis, Fall 2002*

1. Let $f(z) = e^z$. Let $a$ be a positive real number, and let $C$ be the rectangle with vertices $0, a, a + 2\pi i$ and $2\pi i$. Explicitly compute the integral

$$\oint_C f(z)\,dz$$

without using Cauchy's theorem, and verify that Cauchy's theorem applies in this case.




*Solution.* Colloquially, we write

$$\oint_C = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_4}.$$

Now we compute the four integrals individually:

$$\int_{\gamma_1} e^z\,dz = \int_0^a e^t\,dt = e^a - 1.$$

$$\int_{\gamma_2} e^z\,dz = \int_0^1 e^{a(1-t)+(a+2\pi i)t} 2\pi i\,dt = e^a e^{2\pi i t}\Big|_0^1 = 0.$$

$$\int_{\gamma_3} e^z\,dz = \int_0^1 e^{(a+2\pi i)(1-t)+2\pi i t}(-a)\,dt = e^a \int_0^1 e^{-at}(-a)\,dt = e^a e^{-at}\Big|_0^1 = 1 - e^a.$$

$$\int_{\gamma_4} e^z\,dz = \int_0^1 e^{2\pi i(1-t)}(-2\pi i)\,dt = e^{-2\pi i t}\Big|_0^1 = 0.$$

Hence

$$\oint_C f(z)\,dz = (e^a - 1) + 0 + 0 + (1 - e^a) = 0.$$

This verifies Cauchy's Integral Theorem since $f$ is analytic inside the region defined by $C$. $\square$

2. Let $\gamma$ be the semicircular arc from 1 to $-1$ in the upper half plane. Use the $ML$-inequality to prove that

$$\left| \int_\gamma \frac{e^z}{z}\, dz \right| \leq \pi e.$$

*Solution.* The $ML$-inequality tells us in this case that

$$\left| \int_\gamma \frac{e^z}{z}\, dz \right| \leq \max_\gamma \left| \frac{e^z}{z} \right| \cdot L(\gamma),$$

where $L(\gamma)$ is the length of $\gamma$. We know from Kindergarden that $L(\gamma) = \pi$. Now,

$$\max_\gamma \left| \frac{e^z}{z} \right| = \max_\gamma \frac{|e^z|}{|z|} = \max_\gamma |e^z| = \max_\gamma |e^{x+iy}| = \max_\gamma e^x = e.$$

This gives the desired result. $\square$

3. Let $R$ be the region $\mathbb{C} \setminus \{[0, \infty]\}$. Let $f(z) = \sqrt{z}$, considered as a holomorphic function on $R$, and such that $f(-1) = i$.

   (a) Let $\epsilon$ be a small real number, and let $\gamma_\epsilon$ be the path along the unit circle given explicitly by $e^{it}$ for $t \in [\epsilon, 2\pi - \epsilon]$. Compute the integral

$$I_\epsilon = \int_{\gamma_\epsilon} \sqrt{z}\, dz.$$

   *Solution.*

$$\begin{aligned} I_\epsilon &= \int_\epsilon^{2\pi-\epsilon} e^{it/2} \cdot ie^{it}\, dt = i \int_\epsilon^{2\pi-\epsilon} e^{3it/2}\, dt \\ &= (2/3)e^{3it/2}\Big|_\epsilon^{2\pi-\epsilon} = (2/3)(e^{3\pi i - 3i\epsilon/2} - e^{3i\epsilon/2}) \\ &= (2/3)(-e^{-3i\epsilon/2} - e^{3i\epsilon/2}) = -\frac{4}{3}\cos\left(\frac{3\epsilon}{2}\right). \end{aligned}$$

$\square$

   (b) Let $I$ be the limit of the integral as $\epsilon$ approaches 0. Compute $I$, and explain why the fact that $I \neq 0$ does not contradict Cauchy's theorem.

   *Solution.*

$$I = \lim_{\epsilon \to 0} I_\epsilon = \lim_{\epsilon \to 0} -\frac{4}{3}\cos\left(\frac{3\epsilon}{2}\right) = -\frac{4}{3}.$$

   The fact that $I \neq 0$ does not contradict Cauchy's theorem because $f$ is multivalued (and hence not analytic) on any simple curve that encloses the origin. Another way to say this is the branch cut we introduced to make $f$ analytic makes the path for $I$ not closed, so that Cauchy's theorem does not apply. $\square$

(c) Compute the real integral

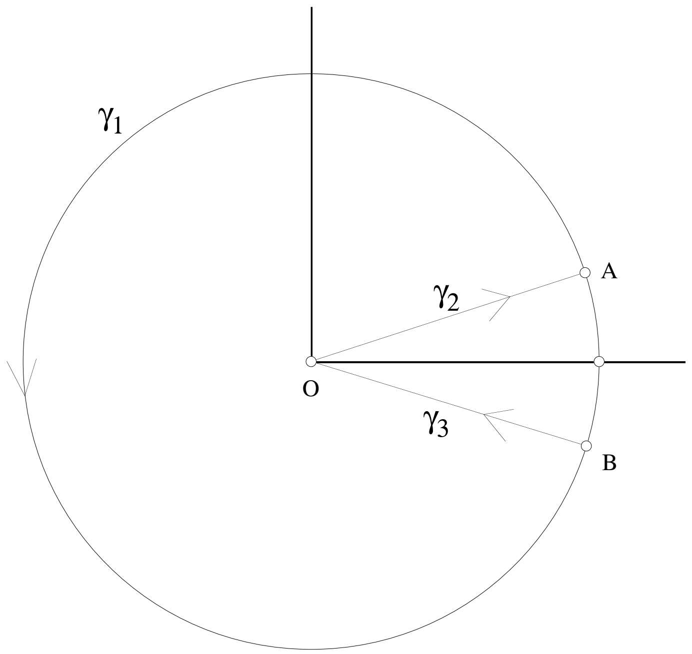
$$J = \int_0^1 \sqrt{x}\, dx.$$

*Proof.* A Math 1a no-brainer:

$$\int_0^1 \sqrt{x}\, dx = \frac{2}{3} x^{3/2} \Big|_0^1 = \frac{2}{3}.$$

□

(d) Prove directly without computing $I$ or $J$ that $I + 2J = 0$.




*Solution.* Let $A = e^{i\epsilon}$ and $B = e^{i(2\pi - \epsilon)}$ in the figure above. Since $\sqrt{0} = 0$ the function $f$ is analytic in the region area enclosed by $\gamma_1 + \gamma_2 + \gamma_3$. By Cauchy's integral formula,

$$\int_{\gamma_1+\gamma_2+\gamma_3} \sqrt{z}\, dz = 0$$

However,

$$\lim_{\epsilon \to 0} \int_{\gamma_1+\gamma_2+\gamma_3} = \lim_{\epsilon \to 0} \left( \int_{\gamma_2} + \int_{\gamma_2} + \int_{\gamma_3} \right) = \lim_{\epsilon \to 0} \int_{\gamma_1} + \lim_{\epsilon \to 0} \int_{\gamma_2} + \lim_{\epsilon \to 0} \int_{\gamma_3} .$$

Now note that $\lim_{\epsilon \to 0} \int_{\gamma_1} = I$ and $\lim_{\epsilon \to 0} \int_{\gamma_2} = J$. It remains to see what $\lim_{\epsilon \to 0} \int_{\gamma_3}$ evaluates to. Since $\gamma_3$ has opposite orientation from $\gamma_2$ this integral gains a minus sign with respect to $J$. However, as $\epsilon \to 0$ the values of $\sqrt{z}$ approach the negative real values of the real square root function. So we pick up a second minus sign. We conlude $\lim_{\epsilon \to 0} \int_{\gamma_3} = J$. Hence $I + 2J = 0$. □

4. If $R$ is a simply connected region with boundary $C$, prove that

$$A = \frac{1}{2i} \oint_C \overline{z}\, dz,$$

where $A$ is the area of $R$.

*Solution.* We apply Green's theorem. Write $\overline{z} = x - iy$, $dz = dx + idy$. All partial derivatives being continuous, we compute

$$\begin{aligned}
\oint_C \overline{z}\, dz &= \oint_C x\, dx + y\, dy + i \oint_C -y\, dx + x dy \\
&= \iint_R (0 - 0) dA + i \iint_R (1 - (-1)) dA \\
&= 2iA,
\end{aligned}$$

from which the desired equality follows immediately. □