Problem Set 2 Solution Set

Anthony Varilly

Math 113: Complex Analysis, Fall 2002

1. Define $\partial f/\partial z$ and $\partial f/\partial \overline{z}$ by setting

$$\partial f/\partial z := \frac{1}{2}(\partial f/\partial x - i \cdot \partial f/\partial y),$$
$$\partial f/\partial \overline{z} := \frac{1}{2}(\partial f/\partial x + i \cdot \partial f/\partial y).$$

Show that the Cauchy–Riemman conditions are equivalent to $\partial f/\partial \overline{z} = 0$. Moreover, show that if $f(z)$ is holomorphic then $f'(z) = \partial f/\partial z$.

Solution. Write $f(z) = u(x,y) + iv(x,y)$, where u and v are real valued. If $\partial f/\partial \overline{z}$ is zero then

$$0 = \partial f/\partial \overline{z} = \frac{1}{2}(\partial u/\partial x + i\partial v/\partial x + i\partial u/\partial y - \partial v/\partial y) \tag{1}$$
$$= \frac{1}{2}(\partial u/\partial x - \partial v/\partial y + i(\partial v/\partial x + \partial u/\partial y)). \tag{2}$$

Notice that the vanishing of the real and imaginary parts of this last expression give precisely the Cauchy–Riemman conditions.

Conversely, suppose the Cauchy–Riemman conditions hold. Then the expression (2) is zero and is equal to $\partial f/\partial \overline{z}$, implying $\partial f/\partial \overline{z} = 0$.

We've seen in class that if $f(z)$ is holomorphic, then

$$f'(z) = \partial u/\partial x + i\partial v/\partial x.$$

We manipulate the definition of $\partial f/\partial z$ using the Cauchy–Riemman equations to obtain

$$\begin{aligned}\partial f/\partial z &= \frac{1}{2}(\partial u/\partial x + i\partial v/\partial x - i\partial u/\partial y + \partial v/\partial y)\\ &= \frac{1}{2}(\partial u/\partial x + \partial u/\partial x + i\partial v/\partial x + i\partial v/\partial x)\\ &= \partial u/\partial x + i\partial v/\partial x = f'(z).\end{aligned}$$

$\square$

2. Let $f(x+iy)$ be a polynomial (with complex coefficients) in x and y. Show that $f(x+iy)$ is holomorphic if and only if it can be expressed as a polynomial in the single variable $z = x+iy$.

Solution. Note that

$$x = \frac{z + \overline{z}}{2} \quad \text{and} \quad y = \frac{z - \overline{z}}{2i},$$

which means that we can rewrite the polinomial $f(x+iy)$ as a polynomial $g(z, \overline{z})$ in the "new" variables z and $\overline{z}$. *Formally*, using the chain rule, we see that

$$\partial g/\partial \overline{z} = \frac{1}{2}(\partial g/\partial x + i \cdot \partial g/\partial y),$$

which coincides with the definition of $\partial g/\partial \overline{z}$ given in the previous problem. If f is analytic, then g should be analytic, yet we know from the Problem 1 that this means $\partial g/\partial \overline{z} = 0$, i.e., g is independent of $\overline{z}$, and hence depends only on z.

Conversely, suppose that f can be expressed as a polynomial in the single variable z. We know that z, considered as a function, is analytic. Since sums, products and scalar multiples of analytic functions are analytic, it follows that f itself is analytic. $\square$

3. Prove or disprove:

$$\lim_{z \to 0} z \cdot \sin\left(\frac{1}{z}\right) = 0.$$

Solution. Let's compute the limit as z approaches 0 from the positive imaginary axis, i.e., $z = ai$ for some $0 < a \in \mathbb{R}$ such that $a \to 0$.

$$\begin{aligned}
\lim_{z \to 0} z \cdot \sin\left(\frac{1}{z}\right) &= \lim_{a \to 0} ai \cdot \sin\left(\frac{1}{ai}\right) \\
&= \lim_{a \to 0} ai \cdot \frac{e^{1/a} - e^{-1/a}}{2i} \\
&= \lim_{a \to 0} a \cdot \frac{e^{1/a} - e^{-1/a}}{2}
\end{aligned}$$

.

This is now a real-variable limit. Since $e^{-1/a} \to 0$ as $a \to 0$, the above limit is equal to

$$\frac{1}{2} \lim_{a \to 0} \frac{e^{1/a}}{1/a}.$$

Using the real-variable rule of L'Hopital, we compute

$$\lim_{a \to 0} \frac{e^{1/a}}{1/a} = \lim_{a \to 0} \frac{(-1/a^2)e^{1/a}}{-1/a^2} = \lim_{a \to 0} e^{1/a} = \infty,$$

and we conlude the claim is false. $\square$

4. Let $f(z)$ and $g(z)$ be holomorphic functions. Prove that the composition $f(g(z))$ is holomorphic, and that its derivative is $f'(g(z))g'(z)$.

Proof. Since f is holomorphic, we have

$$f(y) - f(b) = [f'(y) + \epsilon(y)](y - b),$$

where $\epsilon(y) \to 0$ as $y \to b$. Now substitute $y = g(z)$ and $b = g(a)$ to get

$$f(g(z)) - f(g(a)) = [f'(g(z)) + \epsilon(g(z))](g(z) - g(a)),$$

and so, dividing both sides by $z - a$ we get

$$\frac{f(g(z)) - f(g(a))}{z - a} = [f'(g(z)) + \epsilon(g(z))]\frac{g(z) - g(a)}{z - a}.$$

Since g is differentiable at a, it is also continuous there, so that $\lim_{z\to a} g(z) = g(a) = b$, from which we easily see that $\lim_{z\to a} \epsilon(g(z)) = 0$. Putting it all together:

$$\begin{aligned}
(f \circ g)'(a) &= \lim_{z\to a} \frac{f(g(z)) - f(g(a))}{z - a} \\
&= \left[\lim_{z\to a} f'(g(z)) + \lim_{z\to a} \epsilon(g(z))\right] \lim_{z\to a} \frac{g(z) - g(a)}{z - a} \\
&= f'(g(a))g'(a).
\end{aligned}$$

Hence $f(g(z))$ is holomorphic and its derivative is $f'(g(z))g'(z)$. □

Remark. Some people used Problem 1 to solve this problem. In doing so they appealed to the real-variable chain-rule. That's fine, though tedious to type up and to read. The real-variable proof caries over, so I thought I'd present that.

5. Show that the inverse tangent can be written as

$$\tan^{-1} z = \frac{1}{2i} \log\left(\frac{1 + iz}{1 - iz}\right)$$

Find a branch cut that makes this function holomorphic.

Solution. Let us show that with this definition $\tan(\tan^{-1} z) = z$. Recall that $\tan z = -i(e^{iz} - e^{-iz})/(e^{iz} + e^{-iz})$. We compute

$$\begin{aligned}
\tan(\tan^{-1} z) &= -i\frac{\left(\frac{1+iz}{1-iz}\right)^{1/2} - \left(\frac{1+iz}{1-iz}\right)^{-1/2}}{\left(\frac{1+iz}{1-iz}\right)^{1/2} + \left(\frac{1+iz}{1-iz}\right)^{-1/2}} \\
&= -i\frac{1 + iz - 1 + iz}{1 + iz + 1 - iz} \\
&= -i\frac{2iz}{2} = z.
\end{aligned}$$

The standard branch cut for the logarithm is $(-\infty, 0]$ (i.e., this is the cut that makes $\log z$ analytic in such a way that it coincides with $\log x$ for real x). Let's look for a branch cut of the plane that maps to this half-line under the transformation $z \mapsto (1+iz)/(1-iz)$. Set

$$\frac{1+iz}{1-iz} = -\alpha \quad \alpha \in [0, \infty)$$

Solving for z in terms of α we obtain

$$z = -i\frac{\alpha+1}{\alpha-1}.$$

As α ranges over the non-negative real numbers, z ranges across $(-\infty, -1]i \cup [1, \infty)$. (That by the way, is a horrible abuse of notation.) This is our desired branch cut that will make $\tan^{-1} z$ analytic. $\square$

Remark. Many branch cuts will do the trick. In fact, anything joining the the branch poits i and $-i$ will give a branch cut from 0 to ∞ for the logarithm, thus making the lograithm, and in turn the arctangent, analytic. The one I chose makes the definition of arctangent compatible with that of real variables.

Oh! Also, the branch cut I described above is *not* composed of two branch cuts. Though the cut looks disconnected, we must remember that there is only one point at infinity on the Riemann sphere. So if we say the cut starts at i and moves along the posiive imaginary axis, it will go through ∞ and come back along the negative imaginary axis and end up at $-i$. In fact, were we to draw the above cut on the Riemann sphere, it would look like a segment joining the image of i and $-i$, but passing through the north pole.