Problem Set 10 Solution Set

Anthony Varilly

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1. Given the functional equation

$$\zeta(1-s) = (2\pi)^{-s} \cdot 2 \cdot \cos(\pi s/2)\Gamma(s)\zeta(s),$$

show that $\xi(s) = \xi(1-s)$, where

$$\xi(s) = s(1-s)\Gamma(s/2)\pi^{-s/2}\zeta(s).$$

Solution. We'll look at the quotient $\xi(1-s)/\xi(s)$.

$$\begin{aligned}
\frac{\xi(1-s)}{\xi(s)} &= \frac{s(1-s)\Gamma\left((1-s)/2\right)\pi^{(s-1)/2}\zeta(1-s)}{s(1-s)\Gamma(s/2)\pi^{-s/2}\zeta(s)} \\
&= \frac{\Gamma\left((1-s)/2\right)\pi^{(s-1)/2}(2\pi)^{-s}\cdot 2\cdot\cos(\pi s/2)\Gamma(s)\zeta(s)}{\Gamma(s/2)\pi^{-s/2}\zeta(s)} \\
&= \frac{\Gamma\left((1-s)/2\right)\pi^{-1/2}2^{-s+1}\cos(\pi s/2)\Gamma(s)}{\Gamma(s/2)} \\
&= \frac{\Gamma\left((1-s)/2\right)\cos(\pi s/2)\Gamma(s)}{\Gamma(s/2)\sqrt{\pi}2^{s-1}}
\end{aligned}$$

.

Recall the duplication formula for the Γ function:

$$\frac{\Gamma(2z)}{\Gamma(z)} = \frac{2^{2z-1}\Gamma(z+1/2)}{\sqrt{\pi}}.$$

Substituting $z = s/2$ in the above identity we obtain

$$\frac{\Gamma(s)}{\Gamma(s/2)} = \frac{2^{s-1}\Gamma((s+1)/2)}{\sqrt{\pi}}.$$

Therefore

$$\begin{aligned}
\frac{\xi(1-s)}{\xi(s)} &= \frac{2^{s-1}\Gamma((s+1)/2)}{\sqrt{\pi}} \cdot \frac{\Gamma((1-s)/2)\cos(\pi s)/2}{2^{s-1}\sqrt{\pi}} \\
&= \frac{\Gamma((s+1)/2)\Gamma((1-s)/2)\cos(\pi s)/2}{\pi}.
\end{aligned}$$

We now make use of the identity $\Gamma(z)\Gamma(1-z) = \pi/\sin \pi z$ and set $z = (s+1)/2$ to get

$$\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{1-s}{2}\right) = \frac{\pi}{\sin((s+1)\pi/2)} = \frac{\pi}{\cos \pi s/2}.$$

Finally,

$$\begin{aligned}\frac{\xi(1-s)}{\xi(s)} &= \frac{\Gamma((s+1)/2)\Gamma((1-s)/2)\cos(\pi s)/2}{\pi} \\ &= \frac{\pi}{\cos \pi s/2} \cdot \frac{\cos \pi s/2}{\pi} = 1.\end{aligned}$$

□

2. Prove that

$$\int_x^{x+1} \log \Gamma(z)\, dz = \frac{1}{2}\log(2\pi) + x \log x - x.$$

Solution. We may take the usual branch of the logarithm since the gamma function does not take on negative real values. Thus the function

$$f(x) = \int_x^{x+1} \log \Gamma(z)\, dz$$

is holomorphic. By the fundamental theorem of calculus in the complex setting we have

$$f'(x) = \log \Gamma(x+1) - \log \Gamma(x) = \log\left(\frac{\Gamma(x+1)}{\Gamma}\right).$$

Since the gamma function satisfies the functional equation $\Gamma(x+1) = x\Gamma(x)$ we obtain

$$f'(x) = \log x,$$

from which we conclude that

$$f(x) = x \log x - x + C.$$

To determine the constant C it suffices to evaluate the limit of $f(x)$ as $x \to 0$ from any direction which is not the negative real half-line. On the one hand, a simple application of L'Hopital's rule tells us that

$$\lim_{x \to 0} f(x) = C.$$

On the other hand, Problem 4 from set 8 tells us that

$$\lim_{x \to 0} f(x) = \int_0^1 \log \Gamma(z)\, dz = \frac{1}{2}\log 2\pi.$$

Hence

$$f(x) = \int_x^{x+1} \log \Gamma(z)\, dz = \frac{1}{2}\log(2\pi) + x \log x - x.$$

□

3. Prove that

$$K\left(\frac{1}{\sqrt{2}}\right) := \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-\frac{1}{2}t^2)}} = \sqrt{2}\int_1^\infty \frac{dt}{\sqrt{t^4-1}} = \frac{\Gamma\left(\frac{1}{4}\right)^2}{4\sqrt{\pi}}.$$

Solution. We use the substitution $x^2 = t^2/(2-t^2)$. We have

$$2x\,dx = \frac{4t}{(2-t^2)^2}\,dt \quad \Longrightarrow \quad \frac{dt}{dx} = \frac{x(2-t^2)^2}{2t} = \frac{(2-t^2)^2}{2\sqrt{2-t^2}}.$$

This means the integral $K(1/\sqrt{2})$ transforms as follows:

$$\begin{aligned}\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-\frac{1}{2}t^2)}} &= \sqrt{2}\int_0^1 \frac{1}{\sqrt{(1-t^2)(2-t^2)}} \cdot \frac{(2-t^2)^2}{2\sqrt{2-t^2}}\,dx \\ &= \sqrt{2}\int_0^1 \frac{2-t^2}{2\sqrt{1-t^2}}\,dx.\end{aligned}$$

We want to leave the integral in terms of the variable x only. Note that $t^2 = 2x^2/(1+x^2)$, so $2-t^2 = 2/(1+x^2)$ and $1-t^2 = (1-x^2)/(1+x^2)$. Hence

$$\begin{aligned}\sqrt{2}\int_0^1 \frac{2-t^2}{2\sqrt{1-t^2}}\,dx &= \sqrt{2}\int_0^1 \frac{2}{1+x^2}\sqrt{\frac{1+x^2}{1-x^2}}\frac{dx}{2} \\ &= \sqrt{2}\int_0^1 \frac{dx}{\sqrt{1-x^4}}.\end{aligned}$$

The substitution $t = 1/x$ then gives

$$\sqrt{2}\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \sqrt{2}\int_\infty^1 \frac{(-1/t^2)dt}{\sqrt{1-(1/t)^4}} = \sqrt{2}\int_1^\infty \frac{dt}{\sqrt{t^4-1}}.$$

Now we must show this integral is $\Gamma(1/4)^2/4\sqrt{\pi}$. Consider the substitution $t^2 = \sec\theta$. The associated differential of the substitution is

$$dx = \frac{sin\theta\sqrt{\cos\theta}}{2\cos^2\theta}\,d\theta.$$

Hence

$$\sqrt{2}\int_1^\infty \frac{dt}{\sqrt{t^4-1}} = \sqrt{2}\int_0^{\pi/2} \frac{sin\theta\sqrt{\cos\theta}}{2\cos^2\theta\sqrt{\sec^2\theta-1}}\,d\theta = \sqrt{2}\int_0^{\pi/2}\frac{1}{2\sqrt{\cos\theta}}\,d\theta.$$

The Gamma function we are required to uncover screams we use beta functions; in particular $B(1/4,1/4)$ since this involves $\Gamma(1/4)^2$. So we must somehow relate this integral to $B(1/4,1/4)$. Recall, however that

$$\frac{1}{2}B(m,n) = \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)} = \int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta.$$

So we want to relate our integral to

$$\frac{B(1/4,1/4)}{2} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta\cos\theta}}\, d\theta.$$

Now

$$\begin{aligned}
\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta\cos\theta}}\, d\theta &= \sqrt{2}\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin 2\theta}}\, d\theta \\
&= 2\sqrt{2}\int_0^{\pi/2} \frac{d\theta}{2\sqrt{\sin\theta}}\, d\theta \\
&= 2\sqrt{2}\int_0^{\pi/2} \frac{d\theta}{2\sqrt{\cos\theta}}\, d\theta
\end{aligned}$$

In conclusion,

$$\begin{aligned}
\sqrt{2}\int_1^{\infty} \frac{dt}{\sqrt{t^4-1}} &= \sqrt{2}\int_0^{\pi/2} \frac{d\theta}{2\sqrt{\cos\theta}}\, d\theta \\
&= \frac{B(1/4,1/4)}{4} = \frac{\Gamma(1/4)^2}{4\Gamma(1/2)} \\
&= \frac{\Gamma(1/4)^2}{4\sqrt{\pi}}
\end{aligned}$$

□

4. Evaluate the limit

$$\lim_{s\to 1} \zeta(s) - \frac{1}{s-1}.$$

Solution. We have

$$\begin{aligned}
\zeta(s) = \frac{1}{s-1} + \sum_{n=1}^{\infty}\frac{1}{n^s} - \frac{1}{s-1} &= \frac{1}{s-1} + \sum_{n=1}^{\infty}\frac{1}{n^s} - \int_1^{\infty}\frac{1}{t^s}\, dt \\
&= \frac{1}{s-1} + \sum_{n=1}^{\infty}\left(\frac{1}{n^s} - \int_n^{n+1}\frac{1}{t^s}\, dt\right) \\
&= \frac{1}{s-1} + \sum_{n=1}^{\infty}\int_n^{n+1}\left(\frac{1}{n^s} - \frac{1}{t^s}\right) dt.
\end{aligned}$$

Notice that $\int_n^{n+1}(n^{-s}-t^{-s})\, dt$ is an analytic function for $\operatorname{Re} s > 0$. To show the sum of such integrals (as n ranges from 1 to ∞) is analytic, all we need is convergence on compact sets for which $\operatorname{Re} s > 0$. Now,

$$\left|\int_n^{n+1} n^{-s} - t^{-s}\, dt\right| \le \int_n^{n+1} |n^{-s} - t^{-s}|\, dt \le \sup_{n\le t\le n+1} |\left(n^{-s}-t^{-s}\right)|.$$

This last expression can be bounded as follows. For $n \leq t \leq n+1$, we have

$$\begin{aligned} n^{-s} - t^{-s} &\leq \sup_{n \leq t \leq n+1} \left| \frac{d}{dt} \left(n^{-s} - t^{-s} \right) \right| \\ &= \sup_{n \leq t \leq n+1} \left| \frac{s}{t^{s+1}} \right| \leq \frac{|s|}{n^{1+\mathrm{Re}(s)}}. \end{aligned}$$

The series $\displaystyle\sum_n \frac{1}{n^{1+\operatorname{Re} s}}$ converges uniformly in compact sets for which for $\operatorname{Re} s > 0$. Hence the desired series of integrals converges uniformly in this region as well. Therfore

$$\begin{aligned} \lim_{s \to 1} \zeta(s) - \frac{1}{s-1} &= \lim_{s \to 1} \sum_{n=1}^{\infty} \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{t^s} \right) dt \\ &= \sum_{n=1}^{\infty} \int_n^{n+1} \lim_{s \to 1} \left(\frac{1}{n^s} - \frac{1}{t^s} \right) dt \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{n} - \int_n^{n+1} \frac{dt}{t} \right) \\ &= \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} (\log(n+1) - \log n) \\ &= \lim_{n \to \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log(n+1) \right) = \gamma. \end{aligned}$$

$\square$