Problem Set 1 Solution Set

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1. Solve the following equations in complex numbers, and write down the answer in polar form and rectangular form respectively.

   (a) $z^2 - i = 0$.

   Solution. Write $z = re^{i\theta}$. Then the equation reads $r^2 e^{2i\theta} = e^{i\pi(2n+1)/2}$. The restriction $0 \le \theta \le 2\pi$ gives the solutions $z = e^{i\pi/4}$ and $z = e^{5i\pi/2}$ corresponding to the rectangular forms $\frac{1}{\sqrt{2}}(1+i)$ and $-\frac{1}{\sqrt{2}}(1+i)$, respectively. □

   (b) $z^7 + z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = 0$

   Solution. Note that $(z-1)(z^7+z^6+z^5+z^4+z^3+z^2+z+1) = z^8 - 1$, so the solutions to our equation correspond to the seven non-trivial 8^{th} roots of unity. In polar form, they are $e^{i\pi k/4}$, $k = 1, \ldots, 7$. In rectangular coordinates they are $\frac{1}{\sqrt{2}}(1+i)$, i, $\frac{1}{\sqrt{2}}(-1+i)$, -1, $\frac{1}{-\sqrt{2}}(1+i)$, $-i$ and $\frac{1}{\sqrt{2}}(1-i)$, respectively. □

   (c) $e^{2z} + 2e^z = -2$.

   Solution. We treat the above equation as a quadratic in e^z. By use of the quadratic formula we compute $e^z = -1 \pm i$. Restricting to the principal branch of the logarithm (i.e., where the argument is in the range $(-\pi, \pi)$) we obtain $z = \log(\sqrt{2}) + 3\pi i/4$ and $z = \log(\sqrt{2}) - 3\pi i/4$. □

   (d) $(z+1)/(z-1) = e^{\pi i/3}$.

   Solution. Write $z = a + bi$ and note that $e^{\pi i/3} = (1 + i\sqrt{3})/2$. Hence the equation is equivalent to

   $$(a+1) + bi = (1 + i\sqrt{3})((a-1) + bi)/2.$$

   Equating real and imaginary parts we obtain the system of equations

   $$\begin{aligned} a + 1 &= \frac{a-1}{2} - \frac{b\sqrt{3}}{2}, \\ b &= \frac{(a-1)\sqrt{3}}{2} + \frac{b}{2}. \end{aligned}$$

   Solving this system we compute $a = 0$ and $b = -\sqrt{3}$. Hence $z = -\sqrt{3}i = \sqrt{3}e^{i\pi/2}$. □

Remark. Many people wrote down statements to the effect "the polar form of z is $2\cos\pi + 2\sin\pi$." While the form $r\cos\theta + r\sin\theta$ contains all the necessary information to express z in polar coordinates, it itself is *not* an expression in polar coordinates. The form $r\cos\theta + r\sin\theta$ is actually rectangular once the cosine and sine are evaluated.

2. Let $F(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a polynomial with real coefficients. Suppose that $z = a + bi$ is a root of $F(x)$. Prove that $\overline{z} = a - bi$ is also a root.

   Solution. If $F(z) = 0$, then $\overline{F(z)} = 0$. On the other hand,

$$\begin{aligned}\overline{F(z)} &= \overline{a_0} + \overline{a_1 z} + \cdots + \overline{a_n z^n} \\ &= \overline{a_0} + \overline{a_1} \cdot \overline{z} + \cdots + \overline{a_n} \cdot \overline{z}^n \\ &= a_0 + a_1 \cdot \overline{z} + \cdots + a_n \overline{z}^n = F(\overline{z}).\end{aligned}$$

   Hence $\overline{z}$ is also a root of $F(x)$. □

3. Let $z = a + bi$ with a and b real. What is the real part of $\cos(a + bi)$?

   Solution. Recall that in complex variables the cosine of z is defined by

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

   We easily compute

$$\begin{aligned}\cos(a + bi) &= \frac{e^{i(a+bi)} + e^{-i(a+bi)}}{2} \\ &= \frac{e^{-b}e^{ia} + e^{b}e^{-ia}}{2} \\ &= \frac{e^{-b}(\cos a + i\sin a) + e^{b}(\cos a - i\sin a)}{2} \\ &= \frac{e^{b} + e^{-b}}{2}\cos a - i\frac{e^{b} - e^{-b}}{2}\sin a \\ &= \cosh b \cos a - i \sinh b \sin a.\end{aligned}$$

   Hence the real part of $\cos(a + bi)$ is $\cosh b \cos a$. □

4. Let x and y be any two non-zero complex numbers. If we define $x^y := e^{y \log x}$, then x^y is not well defined due o the ambiguity in the logarithm. What are the possible values of i^i?

   Solution. First, write $i^i = e^{iz}$, where $e^z = i$. Now we solve $e^z = i$. If $z = a + bi$, then $e^z = e^a e^{bi} = e^a(\cos b + i\sin b) = i$. Equating real and imaginary parts we obtain $a = 0$ and $b = (2k + 1/2)\pi$, $k \in \mathbb{Z}$. Hence i^i would take the values $e^{-(2k+1/2)\pi}$, $k \in \mathbb{Z}$. □

5. Let $\omega = e^{2\pi i/p}$ be a p^{th} root of unity. Let

$$\chi(p) = \sum_{n=0}^{p-1} \omega^{n^2}.$$

Show that $\chi(3)^2 = -3$, $\chi(5)^2 = 5$ and $\chi(7)^2 = -7$.

Solution (with an exposition of the general result due to John Provine). Crash and burn. Here we go.

$$\begin{aligned}
\chi(3)^2 &= (1 + 2\omega)^2 = 1 + 4\omega + 4\omega^2 \\
&= -3 + 4(1 + \omega + \omega^2) = -3. \\
\chi(5)^2 &= (1 + 2\omega + 2\omega^4)^2 \\
&= 5 + 4(1 + \omega^2 + \omega^3 + \omega^4) = 5. \\
\chi(7)^2 &= (1 + 2\omega + 2\omega^2 + 2\omega^4)^2 \\
&= 1 + 8\omega + 8\omega^2 + 8\omega^3 + 8\omega^4 + 8\omega^5 + 8\omega^6 \\
&= -7 + 8(1 + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6) = -7.
\end{aligned}$$

Now we prove the more general result

$$\chi(p) = \sqrt{(-1)^{\frac{p-1}{2}} p}.$$

This exposition comes verbatim from John Provine's LaTeX file.

We say that n is a *quadratic residue* modulo a prime p if there exists an integer x such that x^2 is congruent to n modulo p. We will not consider 0 to be a quadratic residue. Let R be the set of quadratic residues modulo p, and let N be the set of quadratic non-residues modulo p. n^2 runs through all the quadratic residues modulo p for $n \in \{1, \dots, \frac{1}{2}(p-1)\}$. Moreover, since $n^2 \equiv (p-n)^2 \mod p$, n^2 runs through each quadratic residue modulo p exactly twice for $n \in \{1, \dots, p-1\}$. Thus

$$\chi(p) = \sum_{n=0}^{p-1} \omega^{n^2} = 1 + 2\sum_{n\in R} \omega^n.$$

But problem 1(b) tells us

$$\sum_{n=0}^{p-1} \omega^n = 1 + \sum_{n\in R} \omega^n + \sum_{n\in N} \omega^n = 0,$$

and so a little subtraction gives

$$\chi(p) = \sum_{n=0}^{p-1} \omega^{n^2} = \sum_{n\in R} \omega^n - \sum_{n\in N} \omega^n.$$

Using the usual Legendre symbol

$$\left(\frac{n}{p}\right) = \begin{cases} 1 & \text{if } n \text{ is a quadratic residue modulo } p \\ -1 & \text{if } n \text{ is a quadratic nonresidue modulo } p \end{cases},$$

we can write our Gauss sum as

$$\chi(p) = \sum_{n=1}^{p-1} \left(\frac{n}{p}\right) \omega^n.$$

Note that if n and m are both quadratic residues or non-residues, then their product is a quadratic residue, and if one is a quadratic residue and one a non-residue, then their product is a quadratic residue. This follows easily from the fact that quadratic residues have even indices and quadratic non-residues have odd indices. Thus the Legendre symbol satisfies the multiplicative property:

$$\left(\frac{n}{p}\right)\left(\frac{m}{p}\right) = \left(\frac{nm}{p}\right).$$

Baring this in mind, the square of the Gauss sum is

$$\begin{aligned} \chi(p)^2 &= \left(\sum_{n=1}^{p-1} \left(\frac{n}{p}\right) \omega^n\right)\left(\sum_{m=1}^{p-1} \left(\frac{m}{p}\right) \omega^m\right) \\ &= \sum_{n=1}^{p-1}\sum_{m=1}^{p-1} \left(\frac{n}{p}\right)\left(\frac{m}{p}\right) \omega^n \omega^m \\ &= \sum_{n=1}^{p-1}\sum_{m=1}^{p-1} \left(\frac{nm}{p}\right) \omega^{n+m}. \end{aligned}$$

Consider the inner sum at some value of n. For each m we can write $m \equiv nr \mod p$ for some r, so that r runs through a complete set of residues modulo p as m varies from 1 to $p-1$. So we can write

$$\chi(p)^2 = \sum_{n=1}^{p-1}\sum_{r=1}^{p-1} \left(\frac{n^2 r}{p}\right) \omega^{n+nr} = \sum_{r=1}^{p-1}\sum_{n=1}^{p-1} \left(\frac{r}{p}\right) \omega^{n(1+r)}.$$

(Note we have changed the order of the summation.) Now the inner sum is

$$\sum_{n=1}^{p-1} \left(\frac{r}{p}\right) \omega^{n(1+r)} = \left(\frac{r}{p}\right) \sum_{n=1}^{p-1} \omega^{n(1+r)} = \begin{cases} p-1 & \text{if } r \equiv -1 \mod p \\ -1 & \text{otherwise} \end{cases}$$

(for if $r \not\equiv -1 \mod p$, then $n(1+r)$ runs through a complete set of residues modulo p, and we have the sum from problem 1(b)). Thus

$$\begin{aligned}\chi(p)^2 &= p\left(\frac{-1}{p}\right) + \sum_{r=1}^{p-1}\left(\frac{r}{p}\right)(-1)\\ &= p\left(\frac{-1}{p}\right)\\ &= \begin{cases} p & \text{if } p \equiv 1 \mod 4\\ -p & \text{if } p \equiv 3 \mod 4\end{cases}\\ &= (-1)^{\frac{p-1}{2}}p.\end{aligned}$$

This proves the result.

□

6. Prove that

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{sin\frac{1}{2}(n+1)\theta \cdot \cos\frac{1}{2}n\theta}{\sin\frac{1}{2}\theta}.$$

Solution. We know that

$$\begin{aligned}\sum_{k=0}^{n} e^{ik\theta} &= \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}\\ &= \frac{-e^{i(n+1)\theta/2}(e^{-i(n+1)\theta/2} - e^{i(n+1)\theta/2})}{-e^{i\theta/2}(e^{-i\theta/2} - e^{i\theta/2})}\\ &= \frac{\sin(n+1)\theta/2}{\sin\theta/2}\cdot e^{in\theta/2}.\end{aligned}$$

Now note that $\sum_{k=0}^{n}\cos k\theta = \mathrm{Re}\left(\sum_{k=0}^{n} e^{ik\theta}\right)$. Hence

$$\begin{aligned}\sum_{k=0}^{n}\cos k\theta &= \mathrm{Re}\left(\frac{\sin(n+1)\theta/2}{\sin\theta/2}\cdot e^{in\theta/2}\right)\\ &= \frac{sin\frac{1}{2}(n+1)\theta\cdot\cos\frac{1}{2}n\theta}{\sin\frac{1}{2}\theta}.\end{aligned}$$

□