Mathematics 113 Fall 2002

MOCK FINAL
Solution Sketches

Note: These solutions are just sketches intended to give enough detail for you to understand what is going on. Certain details and arguments that are omitted here should certainly be included on the exam.

1. Let $F(z)$ be holomorphic function. Suppose that F satisfies the identity

$$F(z) = F(e^z - 1).$$

Prove that F is constant.

Sketch of Proof. Suppose F is not constant. Then at least one of the Taylor series coefficients a_n is non-zero for $n \geq 1$. Write

$$F(z) = a_0 + a_n z^n + a_{n+1} z^{n+1} + \dots$$

with $a_n \neq 0$. Since $e^z - 1 = z + z^2/2 + \dots$ we substitute this into the power series for $F(z)$ to get the power series for $F(e^z - 1)$. Since these are equal, we shall equate the coefficients to obtain a contradiction. We find that $F(z) = F(e^z - 1)$ implies

$$a_0 + a_n z^n + a_{n+1} z^{n+1} + \dots = a_0 + a_n \left(z + z^2/2 + \dots\right)^n + a_{n+1} \left(z + z^2/2 + \dots\right)^{n+1} + \dots$$

Comparing the first few terms we see that

$$a_0 + a_n z^n + a_{n+1} z^{n+1} + \dots = a_0 + a_n z^n + (a_{n+1} + n a_n/2) z^{n+1} + \dots$$

Since the coefficients of z^{n+1} must be the same, we find that $a_n = 0$, a contradiction.

Remark. I would be happy for this problem to be on the exam.

2. Show that the functions $(\sin \sqrt{z}/\sqrt{z})$ and $\sqrt{\sin z/z}$ are holomorphic in a neighbourhood of zero and calculate their radius of convergence.

Sketch of Proof. We see that $\sin \sqrt{z}/\sqrt{z}$ is holomorphic by writing down an explicit power series that equals $\sin \sqrt{z}/\sqrt{z}$ and converges everywhere. Since $\sin z/z$ doesn't vanish at 0, it has a holomorphic square root in a neighbourhood where it doesn't vanish. Since it has a zero at π of order 1, the square root must have a singularity here. The radius of convergence is ∞ and π respectively.

3. Let $P(x, y)$ be a polynomial in two variables with real coefficients. Prove there exists a holomorphic function $f(x + iy) = u(x, y) + iv(x, y)$ with $u(x, y) = P(x, y)$ if and only if

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = 0.$$

Sketch of Proof. The easy direction is on the homework. Suppose that P satisfies the equation. It suffices to construct an appropriate v that satisfies the Cauchy–Riemann equations. Considering $\partial P(x, y)/\partial x$ is a polynomial in y, we may integrate it to obtain a new polynomial $v_1(x, y)$ such that

$$\frac{\partial v_1}{\partial y} = \frac{\partial P}{\partial x}.$$

The polynomial v_1 is defined up to a polynomial in y. Similarly integrate the polynomial $-\partial P(x, y)/\partial y$ with respect to y to obtain $v_2(x, y)$ such that

$$\frac{\partial v_2}{\partial x} = -\frac{\partial P}{\partial y}.$$

The polynomial v_2 is defined up to a polynomial in x. If we can find polynomials $a(y)$ and $b(x)$ such that $v_1 - v_2 = b(x) - a(y)$ we are done, taking $v = v_1 + a(x) = v_2 + b(y)$. On the other hand,

$$\frac{\partial^2(v_1 - v_2)}{\partial x \partial y} = \frac{\partial^2 v_1}{\partial x \partial y} - \frac{\partial^2 v_2}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial v_1}{\partial y} - \frac{\partial}{\partial y}\frac{\partial v_1}{\partial x}$$

$$= \frac{\partial}{\partial x}\frac{\partial P}{\partial x} + \frac{\partial}{\partial y}\frac{\partial P}{\partial y} = \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = 0$$

by construction. The only polynomials F satisfying

$$\frac{\partial^2 F}{\partial x \partial y} = 0$$

are of the form $F(x, y) = b(x) - a(y)$, so we are done.

Remark. This problem is too messy (too many details) to put on the exam.

4. Let n be a non-negative integer. Compute the integral

$$\int_0^\infty \frac{1}{(1 + x^2)^{n+1}}\, dx.$$

Sketch of Proof. The function has a pole of order $n+1$ at i and $-i$. Using the contour in the upper half plane we need only consider i. The integral equals $2\pi i$ times the residue at $x = i$. This is

$$\frac{2\pi i}{n!} \frac{d^n}{dz^n}\bigg|_{z=i} \frac{(z-i)^n}{z^2+1}$$

$$= \frac{2\pi i}{n!} \frac{d^n}{dz^n}\bigg|_{z=i} \frac{1}{(z+i)^n} = \frac{2\pi i}{n!} \left[\frac{(n+1)(n+2)\dots(2n)(-1)^n}{(z+i)^{2n+1}}\right]_{z=i}$$

$$= \frac{2\pi i}{n!} \frac{(n+1)(n+2)\dots(2n)(-1)^n}{(2i)^{2n+1}}$$

$$= \frac{2\pi i(2n)!(-1)^n}{n!^2(2i)^{2n+1}} = \frac{\pi(2n)!}{2^{2n}n!^2}.$$

Remark. Too easy?

5. Prove that

$$\int_0^\infty \frac{\log x}{(1+x^2)^2}\, dx = \frac{-\pi}{4}.$$

Sketch of Proof. Let $f(z) = \log(z)^2/(z^2+1)^2$. Let C be the usual keyhole contour. Take the branch cut of $\log(z)$ that is 0 at 1 above the cut. Then $\log(i) = \pi i/2$ and $\log(-i) = 3\pi i/2$. The outer and inner bits go to zero. The residue of $f(z)$ at i is

$$\frac{d}{dz}\bigg|_{z=i} \frac{\log(z)^2(z-i)^2}{(z^2+1)^2} = \frac{d}{dz}\bigg|_{z=i} \frac{\log(z)^2}{(z+i)^2}$$

$$= \lim_{z\to i} \frac{2\log(z)}{z(z+i)^2} - \frac{2\log(z)^2}{(z+i)^3} = \frac{(\pi i/2)}{2i^3} - \frac{2(\pi i/2)^2}{(2i)^3} = -\pi/4 + i\pi^2/16.$$

The residue of $f(z)$ at i is

$$\frac{d}{dz}\bigg|_{z=-i} \frac{\log(z)^2(z+i)^2}{(z^2+1)^2} = \frac{d}{dz}\bigg|_{z=-i} \frac{\log(z)^2}{(z-i)^2}$$

$$= \lim_{z\to -i} \frac{2\log(z)}{z(z-i)^2} - \frac{2\log(z)^2}{(z-i)^3} = \frac{(3\pi i/2)}{-2i^3} - \frac{2(3\pi i/2)^2}{(-2i)^3} = 3\pi/4 - i9\pi^2/16.$$

Thus

$$\int_0^\infty \frac{(\log z)^2 - (\log z + 2\pi i)^2}{(z^2+1)^2} dz = 2\pi i(\pi/2 - i\pi^2/2).$$

Expanding and dividing by $2\pi i$,

$$-2\int_0^\infty \frac{\log(z)}{(z^2+1)^2} - 2\pi i \int_0^\infty \frac{1}{(z^2+1)^2} = \pi/2 - i\pi^2/2.$$

Equating real and imaginary parts we find that

$$\int_0^\infty \frac{\log(z)}{(z^2+1)^2} = -\frac{\pi}{4}, \qquad \int_0^\infty \frac{1}{(z^2+1)^2} = \frac{\pi}{4}.$$

6. Evaluate the integral

$$\int_0^\infty \sin x^3 \; dx.$$

This argument is almost identical to problem 19, page 184. See also problem 178, page 304.

Sketch of Proof. Let R be real. Let $\zeta = e^{\pi i/6} = (\sqrt{3}+i)/2$, so $\zeta^3 = i$. Let $A = 0$, $B = R$, $C = \zeta R$, and consider the contour that goes from A to B to C to A. By Cauchy's theorem,

$$\oint_C e^{ix^3} = 0.$$

As R goes to infinity the integral from B to C goes to zero. The integral from C to A becomes, letting $x = \zeta t$,

$$\int_R^0 e^{i\zeta^3 x^3} \zeta dx = -\int_0^R \zeta e^{-x^3} dx.$$

as R approaches ∞, letting $x = t^{1/3}$ this becomes

$$-\frac{(\sqrt{3}+i)}{6} \int_0^\infty e^{-t} t^{1/3-1} dt = -\frac{(\sqrt{3}+i)}{2} \cdot \frac{\Gamma(\frac{1}{3})}{3}.$$

Thus we find that

$$\int_0^\infty e^{ix^3} dx = \frac{(\sqrt{3}+i)}{2} \cdot \frac{\Gamma(\frac{1}{3})}{3}.$$

Taking imaginary parts we find that

$$\int_0^\infty \sin x^3 = \frac{\Gamma(\frac{1}{3})}{6}.$$

Remark. Perhaps too difficult?

7. Let f be a holomorphic function such that $|f(z+1)| \leq |f(z)|$ for all $z \in \mathbb{C}$. Suppose moreover that $f(0) = f(1) = 1$. Prove that $f(z+1) = f(z)$.

Sketch of Proof. One sees that $|f(z+1)/f(z)| \leq 1$. Thus by Liouville's theorem, $f(z+1)/f(z)$ is constant. Evaluating at $z = 0$, we find that $f(z+1) = f(z)$.