Mathematics 113 Fall 2002

PRACTICE MIDTERM EXAM
83 Minutes

Requirements: Calculators are not allowed. Show all your working, and submit partial solutions. Write as neatly as possible. Partial marks will be awarded, when appropriate. If you use a theorem from either the text or from class, make sure to state it. Not all questions are worth the same amount. Do not feel you need to write up the questions in order.

1. Let $\zeta = e^{\pi i/2n}$ be a primative $4n$th root of unity.

   (a) Prove that
   $$\zeta + \zeta^3 + \zeta^5 + \ldots + \zeta^{2n-1} = \frac{i}{\sin(\pi/2n)}$$

   (b) Let C be the rectangle with vertices 1, $1+i$, $-1+i$ and -1. Evaluate the following integral:
   $$\oint_C \frac{dz}{z^{2n}+1}.$$

2. Let $f(z)$ be a holomorphic function, and let $u(x, y)$ denote the real part of $f(x+iy)$. Show that
   $$\frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial^2 y} = 0,$$
   justifying all steps.

3. Let $f(z)$ be holomorphic, except for a singularity at $z = 0$.

   (a) Prove that if $1/f(z)$ is bounded in some neighbourhood of zero, then $f(z)$ has a pole of finite order at $z = 0$.

   (b) Suppose that $f(z)$ has an essential singularity at 0. Let $\lambda \in \mathbb{C}$ be any complex number. Show that in any neighbourhood of 0, $f(z) - \lambda$ becomes arbitrarily close to 0.

4. Let $f(z)$ be an entire function such that $f(z+1) = f(z)$.

   (a) By taking the branch cut of $\log z$ along the negative real axis, the function
   $$F(z) := f\left(\frac{\log z}{2\pi i}\right)$$
   becomes holomorphic for all $z \in \mathbb{C} \setminus [-\infty, 0]$. Prove that $F(z)$ extends to a holomorphic function for all $z \neq 0$.

(b) By letting $z = e^{2\pi i \tau}$, show using the Laurent series for $F(z)$ that for all $\tau \in \mathbb{C}$,

$$f(\tau) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i \tau},$$

where

$$a_n = \int_0^1 f(z) e^{-2\pi i z} dz.$$

5. Let $P(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ be a polynomial of degree n. Let R be a region containing all the zeros of $P(x)$, and let C be the boundary of R. Evaluate the integral

$$\oint_C \frac{xP'(x)}{P(x)} dx.$$