Mathematics 113 Fall 2002

MIDTERM EXAM
83 Minutes

Requirements: Calculators are not allowed. Show all your working, and submit partial solutions. Write as neatly as possible. Partial marks will be awarded, when appropriate. If you use a Theorem, make sure to state it. All questions are worth the same amount. Do not feel you need to write up the questions in order.

1. Let f be a holomorphic function, and let $u(x, y)$ be the real part of $f(x+iy)$. Suppose that $u(x, y) = u(-y, x)$. Prove that for all $z \in \mathbb{C}$, $f(z) = f(iz)$.

2. Let $f(z)$ be a holomorphic function such that $f(z)$ is real for $z \in \mathbb{R}$. Prove that for all $z \in \mathbb{C}$,
$$f(\overline{z}) = \overline{f(z)}.$$

3. Let C denote the unit circle in $\mathbb{C}$, and let $f(z)$ be a bounded integrable (but not necessarily holomorphic!) function on C. For all a inside C, let:
$$g(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz.$$
   (a) Prove that $g(a)$ is complex differentiable and thus holomorphic for $|a| < 1$.
   (b) Let $f(z) = 1$ for $\operatorname{Im}\{z\} \geq 0$ and $f(z) = 0$ for $\operatorname{Im}\{z\} < 0$. Explicitly compute $g(a)$ for $|a| < 1$.

4. Let $P(z)$ be a polynomial of degree at least two. Let C be the boundary of a disc containing all the zeros of $P(z)$. Prove that
$$\oint_C \frac{dz}{P(z)} = 0.$$

5. (a) What are the complex zeros of $\cos z - \sin z$?
   (b) What is the radius of convergence of the Taylor series of
$$f(z) = \frac{1}{\cos z - \sin z}$$
   around $z = 0$?