Mathematics 113 Fall 2002

HOMEWORK ASSIGNMENT # 7
DUE, Thursday, November 14

Collaboration: On the homework sets, collaboration is not only allowed but encouraged. However, you must write up and understand your own individual homework solutions, and you may not share written solutions. If you learn how to solve a problem by talking to a classmate, CA, or looking it up in a book, you should cite the source in your homework write-up, just as you would reference your sources in a literature or history class.
Show all your working, and write up your solutions as neatly as possible.

1. Let $P(z)$ be a polynomial. Prove there exists an real positive number ϵ with the following property: for all non-zero complex numbers $|\lambda| < \epsilon$, the polynomial $P(z)+\lambda$ has distinct roots.

2. Prove that if m is a real number such that $m > 0$, then
$$\int_0^\infty \frac{\cos mx}{(x^2+1)^2} dx = \frac{\pi e^{-m}(1+m)}{4}.$$

3. If $a^2 > b^2 + c^2$, prove that
$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta + c\sin\theta} = \frac{2\pi}{\sqrt{a^2 - b^2 - c^2}}.$$

4. Let $\lambda = a + bi$ with $a > 0$.
   (a) Evaluate the integral
   $$\int_{-\infty}^\infty \frac{x \sin x}{x^2 + \lambda^2} dx.$$
   (b) Using your answer to part (a), prove that
   $$\int_{-\infty}^\infty \frac{x \sin x}{x^4 + 4a^4}\, dx = \frac{\pi e^{-a} \sin a}{2a^2}.$$

5. Let p and q be integers with $q > p > 0$.
   (a) Evaluate the following integral
   $$\int_{-\infty}^\infty \frac{t^{2p}}{t^{2q}+1} dt.$$
   (b) Use part (a) to evaluate the integral
   $$\int_0^\infty \frac{x^{s-1}}{1+x} dx,$$
   when $0 < s = (2p+1)/2q < 1$.

ALTERNATE ASSIGNMENT

If you choose to do this assignment, it will be due *in class* on Thursday (I will be marking it). For integral k, evaluate the integral

$$S_k := \int_{-\infty}^{\infty} \frac{\sin^k x}{x^k}\, dx.$$

in a simple an expression as you can. What answer do you get when $k = 17$? Prove that S_k is always a rational multiple of π. See if you can you prove that

$$\lim_{k\to\infty} \sqrt{k} \cdot S_k = \sqrt{6\pi}.$$