HOMEWORK ASSIGNMENT # 4
DUE, Thursday, October 17

Collaboration: On the homework sets, collaboration is not only allowed but encouraged. However, you must write up and understand your own individual homework solutions, and you may not share written solutions. If you learn how to solve a problem by talking to a classmate, CA, or looking it up in a book, you should cite the source in your homework write-up, just as you would reference your sources in a literature or history class.
Show all your working, and write up your solutions as neatly as possible.

1. (a) Let n be a positive integer. Using Cauchy's Integral Formula, calculate the integral
$$\oint_C \left(z - \frac{1}{z}\right)^n \frac{dz}{z}$$
where C is the unit circle in $\mathbb{C}$.

   (b) By using the substitution $z \mapsto e^{it}$ in the integral above, evaluate
$$\int_0^{2\pi} \sin^n z \, dz.$$

2. Let τ be a complex number that is not real. Let $f(z)$ be a holomorphic function such that $f(z+1) = f(z)$ and $f(z+\tau) = f(z)$. Prove that $f(z)$ is constant. (Hint: Use Liouville's Theorem).

3. Let $P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$, where $a_n \neq 0$. Show there exists n complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$, possibly not distinct, such that
$$P(z) = a_n (z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n)$$

4. Let C be the circle $|z| = 2$ in $\mathbb{C}$. Evaluate the integral

$$\frac{1}{2\pi i} \oint_C f(z) dz$$

for the following functions $f(z)$. Here λ denotes some complex number and $k \in \mathbb{N}$. Don't submit your working for this question.

(a) $(1+z)/z^2$
(b) $1/(z-a)$, $|a| < 2$
(c) $1/(z-a)$, $|a| > 2$
(d) $\sin z/z$
(e) $\sin z/z^2$
(f) $1/(1+z^2)$
(g) $e^{\lambda z}/(1+z^2)$
(h) $\cos \pi z/(z^2-1)$
(i) $\cos^2 \lambda z/z^3$
(j) $\cos z/\sin z$
(k) $1/(e^z-1)$
(l) $(e^z-1)/\sin^2 z$
(m) $(z-\sin z)/z^2 \sin z$
(n) e^z/z^k
(o) $e^{1/z}$
(p) $e^{1/z}/(1-z)$
(q) $1/(z^2+z+1)$
(r) $\cos 2\pi z/(z^2+z+1)$
(s) $1/(z^2+4z+5)^2$
(t) $1/(z^k-1)$
(u) $\tan z/z^2$
(v) $\tan z/z^{2k+1}$