Solution Set 8

Math 113
December 26, 2001

1. *Derive the fundamental theorem of algebra as a consequence of Rouché's theorem*

Let $f(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_0$ with $n \geq 1$. As discussed before, it suffices to find a zero of f.

$$\lim_{z\to\infty} \frac{a_n z^n}{f(z) - a_n z^n} = \infty$$

because the degree of the numberator is greater than the degree of the denominator. This implies that we have a r such that for $|z| \geq r$ $\frac{|a_n z^n|}{|a_{n-1}z^{n-1}+\ldots a_0|} > 1$. So on $|z| = r$, $|a_n z^n| > |a_{n-1}z^{n-1} + \ldots + a_0|$. We then apply Rouché's theorem to the circle of radius r, to see that the number of zeros inside the disk of radius r of f is n, i.e. the number of zeros of $a_n z^n$ inside the disk of radius r.

2.

Find the number of zeros of

a) $f(z) = e^z - 2z - 1$ *in* $|z| \leq 1$

We claim that on the unit circle, $|2z + 1| > |e^z|$ which will imply by Rouchè's theorem that f has one zero in the unit disk. So we establish the claim. $|e^z| = |e^x|$. $|2z + 1| = |(2x + 1) + 2i\sqrt{1 - x^2}| = \sqrt{4x + 5}$. The claim holds when $x = -1$ and $x = 1$ by inspection. Furthermore, as real functions, e^x is convex and $\sqrt{4x + 5}$ is concave (by, for instance, taking second derivatives), so the claim holds and the answer is 1.

b) $f(z) = z^6 + 9z^4 + z^3 + 2z + 4$ *in* $|z| \leq 1$

On the unit circle, $|9z^4| = 9 > 1+1+2+4 = |z^6|+|z^3|+|2z|+4 \geq |z^6+z^3+2z+4|$, so Rouché's theorem says that the number of zeros of f in the unit disk is the same as the number of zeros of $9z^4$, namely 4.

c) $f(z) = z^4 - 5z + 1$ *in* $1 \leq |z| \leq 2$

For $|z| = 2$, $|z^4| = 16 > |-5z + 1|$, so Rouché's theorem tells us that the number of zeros of f in $|z| < 2$ is the same as the number of zeros of z^4 in that region, namely 4. f has no roots on the circle of radius 2 by the triangle inequality (i.e. there is no triangle with sides of norms 4,1, and 10). On $|z| = 1$, $|-5z| = 5 > |z^4 + 1|$, so by Rouché's theorem there is 1 zero of f in $|z| < 1$. There are therefore $4 + 0 - 1 = 3$ zeros of f in $1 \leq |z| \leq 2$.

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3. *Suppose f is entire and that $f(z)$ is real iff z is real. Use the argument principle to show that f can have at most one zero.*

It is enough to show that there is at most one zero of f in a disk of radius r for all r. Fix r. Let γ be the circle of radius r. Let γ_1 be the intersection of γ with the upper half plane. $f(\gamma_1)$ does not touch the real axis and it therefore contained in either the upper or the lower half planes. This implies that the change in argument over $f(\gamma_1)$ is less than π. Similarly we may let γ_2 be the intersection of γ with the lower half plane, and we have that the change in argument over $f(\gamma_2)$ is less than π. This implies that the number of zeros minus the number of poles of f inside γ is $\frac{1}{2\pi}\Delta \mathrm{Arg} f(\gamma) < \frac{1}{2\pi}(2\pi) = 1$. Since f has no poles, the result follows.

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4. a) *Let k be an integer. Prove directly that $f(z) = z^k$ is locally injective for $z \neq 0$.*

If $f(z_1) = f(z_2)$ with $z_1 \neq z_2$, $|z_1|^k = |z_2|^k$, which implies that $|z_1| = |z_2|$, and $k\mathrm{Arg}(z_1) - k\mathrm{Arg}(z_2)$ is divisible by 2π, which implies that $\mathrm{Arg}(z_1) - \mathrm{Arg}(z_2)$ is greater than $2\pi/k$. So on an open pie slice $\{z|\theta < \arg(z)\theta + \frac{2\pi}{k}\}$ for any θ and some appropriate choice of argument, f is injective. Since these regions cover, f is locally injective.

b) *Find the largest disk centered at $z = 1$ on which the function $f(z) = z^4$ is injective.*

By the same reasoning as in the first part, f is injective on the region between the two lines $y = x$ and $y = -x$. Plane geometry then shows that the radius of the largest disk centered at $z = 1$ on which f is injective is $\frac{1}{\sqrt{2}}$.

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5. *Near what points are the following maps conformal?*

a) $f(z) = \bar{z}$

f is analytic nowhere, and it is C^1, so by the discussion from class we know that f is conformal nowhere. To see this directly, choose a point $z_0 = x_0 + iy_0$ and let L_1 be the horizontal line through z_0 and L_2 be the vertical line through z_0 parameterized by arc length, $L_1(t) = x_0 + t + iy_0$, $L_2(t) = x_0 + i(y_0 + t)$. The angle from the tangent vector of L_1 to the tangent vector of L_2 at z_0 is $\pi/2$. $f(L_1(t)) = x_0 + t - iy_0$. $f(L_2(t)) = x_0 - iy_0 - it$. The angle from the tangent vector of $f(L_1(t))$ to the tangent vector of $f(L_2(t))$ at $f(z_0)$ is $3\pi/2 = -\pi/2$. Since these angles are not the same, f is nowhere conformal.

b) $f(z) = tan(z)$

f is defined on $\mathbf{C} - \{\pi/2 + k\pi | k \in \mathbf{Z}\}$ because as determined on a previous problem set the zeros of cosine are $\{\pi/2 + k\pi | k \in \mathbf{Z}\}$. On this domain however, f is analytic and $f'(z) = \frac{1}{(\text{Cos } z)^2} \neq 0$, which as discussed in class implies that f is conformal on this domain.

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6. *What is the image of a horizontal or vertical line in* $\mathbf{C}$ *under the mapping* $f(z) = e^z$*?*

Use the notation L_1 and L_2 from the previous problem. $f(L_1(t)) = e^{z_0}e^t$ with $t \in \mathbf{R}$. Since e^{z_0} is a constant and e^t ranges over the interval $(0, \infty)$, $f(L_1(t))$ traces out a ray from, and not including, the origin, with the same argument and e^{z_0}. $f(L_2(t)) = e^{z_0}e^{it}$. Since e^it traces out a circle, so does $f(L_2(t))$. $f(L_2(t))$ is the circle with radius e^{x_0} centered at the origin.

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7. *The Fourier transform of a function* $f : \mathbf{R} \to \mathbf{C}$, *if it exists, is a function* $\hat{f} : \mathbf{R} \to \mathbf{C}$ *defined by the improper integral*

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-iwx}dx.$$

The normal distribution probability function is given by the formula

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}.$$

Prove that f *is equal to its own Fourier transform, i.e. that the improper integral defining* f *exists, and that* $\hat{f}(w) = f(w)$ *for all* $w \in \mathbf{R}$. *(You may use the standard fact from real analysis that* $\int_{-\infty}^{\infty} e^{-x^2/2}dx = \sqrt{2\pi}$.*) [**Hint**: Prove that* $\int_{-\infty}^{\infty} e^{-(x+iw)^2/2}dx = \sqrt{2\pi}$ *for all* $w \in \mathbf{R}$ *by considering the contour integral of* $g(z) = e^{-z^2/2}$ *around a rectanlge* C *with vertices at* $-R, R, R + wi, -R + wi$ *and letting* $R \to \infty$*]*

We establish that $\int_{-\infty}^{\infty} e^{-(x+iw)^2/2}dx = \sqrt{2\pi}$ for all $w \in \mathbf{R}$. To show that this limit in fact exists you can expand out $-(x + iw)^2$, note that $e^{w^2/2}$ is a constant, and express e^{-ixw} in terms of sines and cosines to see that the integral converges. Here we'll get the convergence in the process of the computation as most people wanted to do. The annoying little point is just that the existance of $\lim_{R\to\infty} \int_{-R}^{R} g(x)dx$ does not imply the existance of $\int_{-\infty}^{\infty} g(x)dx$ defined by $\lim_{R_1\to\infty} \lim_{R_2\to\infty} \int_{-R_1}^{R_2} g(x)dx$. So if we do existence and computation at the same time we change our rectangle to C, the rectangle with vertices $-R_1, R_2, R_2 + wi, -R_1 + wi$. Call the segments from $-R_1 + wi$ to $-R_1$ and R_2 to $R_2 + wi$, V_1 and V_2 respectively.

$$\int_C g(z)dz = \int_{V_1} g(z)dz + \int_{V_2} g(z)dz + \int_{-R_1}^{R_2} e^{-x^2/2}dx - \int_{-R_1}^{R_2} e^{-(x+iw)^2/2}dx.$$

By the closed curce theorem, $\int_C g(z)dz = 0$, which implies

$$\int_{-R_1}^{R_2} e^{-(x+iw)^2/2}dx = \int_{V_1} g(z)dz + \int_{V_2} g(z)dz + \int_{-R_1}^{R_2} e^{-x^2/2}dx.$$

$\int_{V_j} g(z)dz \le w\mathrm{Max}|e^{-(R_j+iy)^2/2}| = w\mathrm{Max}|e^{-(R_j)^2/2}||e^{y^2/2}|$. This implies that $\int_{V_j} g(z)dz$ approaches 0 as $R_j \to \infty$. Now take the limit of $\int_{-R_1}^{R_2} e^{-(x+iw)^2/2}dx$ as $R_2 \to \infty$. Because $\int_{-\infty}^{\infty} e^{-x^2/2}dx$ exists $\int_{-R_1}^{\infty} e^{-x^2/2}dx$ exists. $\int_{V_2} g(z)dz$ approaches 0, so this exists as well, and we conclude that $\lim_{R_2\to\infty} \int_{-R_1}^{R_2} e^{-(x+iw)^2/2}dx$ exists. Similar reasoning shows that the limit of $\lim_{R_2\to\infty} \int_{-R_1}^{R_2} e^{-(x+iw)^2/2}dx$ as $R_1 \to \infty$ exists and we have

$$\int_{-\infty}^{\infty} e^{-(x+iw)^2/2}dx = \int_{-\infty}^{\infty} e^{-x^2/2}dx = \sqrt{2\pi}$$

because the other two terms vanish in the limit.

Existence of $\int_{-\infty}^{\infty} e^{-(x+iw)^2/2}dx$ implies the existence of $\hat{f}$ because the integrals differ only by a constant. Furthermore,

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} e^{-iwx} dx = \frac{1}{2\pi} e^{-w^2/2} \int_{-\infty}^{\infty} e^{-(x+iw)^2/2}dx = \frac{1}{\sqrt{2\pi}} e^{-w^2/2} = f(w),$$

as desired.