Math 113, Fall 2001

Solutions to Problem Set 7

December 4, 2001

1a. $z\cot z$ has poles at $z_0 = n\pi$, $n \in \mathbf{Z} - 0$. Remember that $z_0 = 0$ is a removable singularity. Now $\cot z = \frac{\cos z}{\sin z}$, where $\cos z = 1 - \frac{z^2}{2} + \ldots$ and $\sin z = z - \frac{z^3}{3!} + \ldots = z(1 - \frac{z^2}{3!} + \ldots)$. Hence $\cot z = z^{-1}(1 + O(z^2))$, where $O(z^2)$ denotes terms of order 2 or higher in z. Therefore, $\cot z$ has a simple pole at $z_0 = 0$ with residue 1. Since $\cot z = \cot(z - n\pi)$, $\cot z$ has residue 1 at $z_0 = n\pi$. Hence $z\cot z$ has residue $n\pi$ at the singularity $z_0 = n\pi$ for all $n \in \mathbf{Z} - 0$.

1b. $z^4 - 1 = (z-1)(z-i)(z+1)(z+i)$. Thus $z_0 = 1$ is a simple pole, while $z_0 = i, -1, -i$ are double poles. Using the formulas given in problem 3, one calculates the residues at $1, i, -1, -i$ to be $\frac{1}{16}$, $\frac{2+3i}{16}$, $-\frac{3}{16}$ and $\frac{2-3i}{16}$ respectively.

1c. The only singularity of $\sin\frac{1}{z}$ is at $z_0 = 0$. Since $\sin z = z - \frac{z^3}{3!} + \ldots$ in a neighborhood of 0, we can write $\sin\frac{1}{z} = z^{-1} - \frac{z^{-3}}{3!} + \ldots$ in a deleted neighborhood of 0. Since Laurent expansions are unique, this series must be the Laurent expansion for $\sin\frac{1}{z}$. Hence the residue at $z_0 = 0$ is 1.

1d. The singularities of $\frac{1}{e^z - 1}$ are at $z_0 = 2\pi in$ for all $n \in \mathbf{Z}$. Near 0, $e^z - 1 = z + \frac{z^2}{2} + \ldots = z(1 + O(z))$, so $\frac{1}{e^z-1} = z^{-1}(1 + O(z))$ and the residue at $z_0 = 0$ is 1. Since $\frac{1}{e^z-1}$ is periodic with period $2\pi in$, the residue at each pole $2\pi in$ is 1.

2. Note that $e^{\frac{z-1}{z}} = e^{1-\frac{1}{z}} = e(e^{-\frac{1}{z}})$, so the only singularity is at 0. Now $e^{-\frac{1}{z}} = 1 - z^{-1} + \frac{z^{-2}}{2} + \ldots$, so one sees that $(2z-1)e^{\frac{z-1}{z}} = 2ez - 3e + 2ez^{-1} + \ldots$, so the residue at $z_0 = 0$ is 2. Hence the integral is $4e\pi i$.

3a. If $A(z)$ has a zero of order k at $z0$, then we may write

$$A(z) = \frac{A^{(k)}(z_0)}{k!}(z - z_0)^k(1 + O(z - z_0)).$$

Similarly we may write

$$B(z) = \frac{B^{(k+1)}(z_0)}{(k+1)!}(z - z_0)^{k+1}(1 + O(z - z_0)).$$

Therefore,

$$\frac{A(z)}{B(z)} = (k+1)\frac{A^{(k)}(z_0)}{B^{(k+1)}(z_0)}(z-z_0)^{-1}(1+O(z-z_0)),$$

so z_0 has a simple pole at z_0, and the residue of $\frac{A(z)}{B(z)}$ at z_0 is $(k+1)\frac{A^{(k)}(z_0)}{B^{(k+1)}(z_0)}$.

Now z^3 has a zero of order 3 at $z_0 = 0$. $1-\cos(z) = -\frac{z^2}{2}(1+O(z^2))$, so $(1-\cos(z))^2 = \frac{z^4}{4}(1+O(z^2))$, so $(1-\cos(z))^2$ has a zero of order 4 at $z_0 = 0$, and there are no other zeroes in the unit circle. The third derivative of z^3 is 6, while the fourth derivative of $(1-\cos(z))^2$ is $-2\cos z + 8\cos 2z$. Hence the residue of $\frac{z^3}{(1-\cos(z))^2}$ at $z_0 = 0$ is 4, and the integral is $8\pi i$.

3b. We may write

$$A(z) = A(z_0) + A'(z_0)(z-z_0) + O((z-z_0)^2)$$

and

$$\begin{aligned} B(z) &= \frac{B''(z_0)}{2}(z-z_0)^2 + \frac{B'''(z_0)}{3!}(z-z_0)^3 + O((z-z_0)^4) \\ &= \frac{B''(z_0)}{2}(z-z_0)^2(1+\frac{1}{3}\frac{B'''(z_0)}{B''(z_0)}(z-z_0) + O((z-z_0)^2)). \end{aligned}$$

Therefore

$$\frac{1}{B(z)} = \frac{2}{B''(z_0)}(z-z_0)^{-2}(1-\frac{1}{3}\frac{B'''(z_0)}{B''(z_0)}(z-z_0) + O((z-z_0)^2)).$$

Thus $\frac{A(z)}{B(z)}$ has a double pole at z_0, and its residue is the coefficient of the $(z-z_0)^{-1}$ term, which is

$$\frac{2}{B''(z_0)}(A'(z_0) - A(z_0))\frac{B'''(z_0)}{3B''(z_0)}) = 2\frac{A'(z_0)}{B''(z_0)} - \frac{2A(z_0)B'''(z_0)}{3(B''(z_0))^2}.$$

To apply this result to the integral, we note that the value of e^z and its first derivative at $z_0 = 1$ are both e. The value of the second derivative of $(z-1)^2$ at $z_0 = 1$ is 2, while the third derivative vanishes. One then computes the residue of $\frac{e^z}{(z-1)^2}$ at $z_0 = 1$ to be e, so the integral is $2\pi i e$.

4a. We evaluate the integral of $\frac{z^2}{1+z^4}$ around a semicircular contour of radius R. The integral from $z = -R$ to $z = R$ gives us $\int_{-R}^{R}\frac{x^2 dx}{1+x^4}$, while the integral over the semicircular arc approaches 0 as R approaches infinity. Hence we need only calculate the residues of $\frac{z^2}{1+z^4}$ in the upper half plane.

Put $\zeta = e^{\frac{i\pi}{8}} = \frac{1}{\sqrt{2}}(1+i)$. Observe that

$$1+z^4 = (z-\zeta)(z-\zeta^3)(z-\zeta^5)(z-\zeta^7).$$

Only ζ and ζ^3 lie in the upper half plane. The residue of $\frac{z^2}{1+z^4}$ at ζ is $\frac{\zeta^2}{(\zeta-\zeta^3)(\zeta-\zeta^5)(\zeta-\zeta^7)}$, and the residue at ζ^3 is $\frac{\zeta^6}{(\zeta^3-\zeta)(\zeta^3-\zeta^5)(\zeta^3-\zeta^7)}$. The sum of the residues equals $-\frac{i}{2\sqrt{2}}$, so the integral is $\frac{\pi}{\sqrt{2}}$.

4b. Put $z = e^{i\theta}$. Then $dz = ie^{i\theta}d\theta = izd\theta$, so $d\theta = -\frac{i}{z}dz$. Also, $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}(z - \frac{1}{z})$, so $1 + \sin^2\theta = 1 - \frac{1}{4}(z^2 - 2 + \frac{1}{z^2})$, and we get that $\int_0^{2\pi} \frac{d\theta}{1+\sin^2\theta} = \int_{|z|=1} \frac{4iz}{z^4-6z^2+1}$. We can write

$$z^4 - 6z^2 + 1 = (z - (1 + \sqrt{2}))(z - (1 - \sqrt{2}))(z - (-1 + \sqrt{2}))(z - (-1 - \sqrt{2})),$$

so the poles inside the unit circle are just $z_0 = \pm(1 - \sqrt{2})$. One calculates the sum of the residues to be $-\frac{i}{\sqrt{2}}$, so the integral is $\sqrt{2}\pi$.

4c. Put $\zeta = e^{\frac{i\pi}{n}}$. We integrate along a pie wedge of angle $\frac{2\pi}{n}$ and radius R, starting out along the positive x-axis for R units, then making an arc of angle $\frac{2\pi}{n}$, then coming back in a radial line towards the origin. The integral of $\frac{dz}{1+z^n}$ along this contour will give us $\int_0^R \frac{dx}{1+x^n}$ along the positive x-axis, a piece that dies off as $R \to \infty$, and $-\zeta^2 \int_0^R \frac{dz}{1+z^n}$ along the other edge of the pie wedge. Hence the desired integral $\int_0^\infty \frac{dx}{1+x^n}$ equals the integral of $\frac{dz}{1+z^n}$ around the contour, as $R \to \infty$, divided by $(1 - \zeta^2)$.

The only pole of $\frac{1}{1+z^n}$ inside the pie wedge is at $z = \zeta$. Now $(1 + z^n) = (z - \zeta)(z^{n-1} + z^{n-2}\zeta + \cdots + \zeta^{n-1})$, so the residue of $\frac{1}{1+z^n}$ at ζ is $\frac{1}{n\zeta^{(n-1)}} = -\frac{\zeta}{n}$. Hence the contour integral equals $(2\pi i)(-\frac{\zeta}{n})$. The integral $\int_0^\infty \frac{dx}{1+x^n}$ equals $(2\pi i)(-\frac{\zeta}{n})(\frac{1}{1-\zeta^2}) = \frac{2\pi i}{n}\frac{1}{\zeta-\zeta^{-1}} = \frac{\frac{\pi}{n}}{\sin(\frac{\pi}{n})}$.

5a. Following Bak and Newman, we consider the integral $\int_{C_N} \frac{\pi\cot\pi z}{z^2+a^2}$, where for each positive integer N, C_N is the square passing through the points $\pm(N + \frac{1}{2})$ and $\pm i(N + \frac{1}{2})$. The estimates in Bak and Newman show that as N approaches infinity, the integral approaches zero. Now the poles of $\frac{\pi\cot\pi z}{z^2+a^2}$ are at $z_0 = n$ with residue $\frac{1}{n^2+a^2}$, and at $z_0 = \pm ai$. One easily checks that the residue at $\pm ai$ is $\frac{\pi\cot(\pm a\pi i)}{\pm 2ai}$. The sum of the residues at $\pm ai$, after some manipulation, is $-\frac{\pi\coth a\pi}{a}$. Since the contour integral is zero, the sum of the residues at $\pm ai$ is equal to the negative of the sum of the residues at each $z_0 = n$, so the sum of the residues at $z_0 = n$ is $\frac{\pi\coth a\pi}{a}$. The sum that we want, $\sum_{n=0}^\infty \frac{1}{n^2+a^2}$ is simply $\frac{1}{2}(\frac{\pi\coth a\pi}{a}) + \frac{1}{2a^2}$.

5b. Again following Bak and Newman, we observe that $\binom{3n}{n}$ is the coefficient of z^n in $(1 + z)^{3n}$, so it is the residue at $z_0 = 0$ of the function $\frac{(1+z)^{3n}}{z^{n+1}}$. Hence for any circle C centered at 0, we may write $\binom{3n}{n} = \frac{1}{2\pi i}\int_C \frac{(1+z)^{3n}}{z^{n+1}}$. We can then write $\binom{3n}{n}8^{-n} = \frac{1}{2\pi i}\int_C \frac{(1+z)^{3n}}{(8z)^n}\frac{dz}{z}$, so $\sum_{n=0}^\infty \binom{3n}{n}8^{-n} = \sum_{n=0}^\infty \frac{1}{2\pi i}\int_C \frac{(1+z)^{3n}}{(8z)^n}\frac{dz}{z}$. To push the summation under the integral, we need $\sum_{n=0}^\infty \frac{1}{z}\frac{(1+z)^{3n}}{(8z)^n}$ to converge uniformly on C. This can be accomplished by taking C to be the circle of radius $\frac{1}{2}$ centered at 0; on this circle $|\frac{(1+z)^{3n}}{(8z)^n}|$ is bounded by $\frac{27}{32} < 1$.

Switching the sum and integral, we get that

$$\sum_{n=0}^{\infty} \binom{3n}{n} 8^{-n} = \frac{1}{2\pi i} \int_C \sum_{n=0}^{\infty} \frac{dz}{z} \frac{(1+z)^{3n}}{(8z)^n}$$
$$= \frac{1}{2\pi i} \int_C \frac{1}{z(1 - \frac{1+3z+3z^2+z^3}{8z})}$$
$$= \frac{1}{2\pi i} \int_C \frac{-8}{z^3 + 3z^2 - 5z + 1}.$$

One checks that $z^3 + 3z^2 - 5z + 1 = (z-1)(z-(-2+\sqrt{5}))(z-(-2-\sqrt{5}))$. The only pole inside the circle C of radius $\frac{1}{2}$ is $z_0 = -2+\sqrt{5}$, and the residue of $\frac{-8}{z^3+3z^2-5z+1}$ ar $z_0 = -2+\sqrt{5}$ is $\frac{-8}{(-3+\sqrt{5})(2\sqrt{5})}$. Hence $\sum_{n=0}^{\infty} \binom{3n}{n} 8^{-n} = \frac{4}{3\sqrt{5}-5}$.

6. Under the assumptions given, in a deleted neighborhood of z_0 we may write $f(z) = \frac{A}{z} + g(z)$, where A is the residue of f at z_0 and g is analytic in the entire neighborhood of z_0. Now given any $\epsilon > 0$, $|g(z)|$ is bounded by ϵ on the arc γ_r for sufficiently small r. We have that, $\int_{\gamma_r} f = \int_{\gamma_r} \frac{A}{z} + \int_{\gamma_r} g$. Thus for sufficiently small r, the integral $\int_{\gamma_r}$ is bounded above by $\epsilon\alpha r$ by the ML-inequality. The other piece $\int_{\gamma_r} \frac{A}{z}$ is always equal to $A\alpha i$. Hence taking the limit as $r \to 0$, we get $\lim_{r\to 0} \int_{\gamma_r} f = A\alpha i = \alpha i \mathrm{Res}(f; z_0)$.

7a. If $f(z) = \frac{1}{z}$, then $-\frac{1}{w^2} f(\frac{1}{w}) = -\frac{1}{w}$. So f has a removable singularity at $z = \infty$ (it can be defined to be 0 there), but it has residue -1 at $z = \infty, w = 0$.

7b. Making the change of variables $w = \frac{1}{z}$, we have

$$\int_{|z|=R} f(z)dz = -\int_{|w|=\frac{1}{R}} -\frac{1}{w^2} f(\frac{1}{w}).$$

Since $R > R_0$, the only possible pole of $-\frac{1}{w^2} f(\frac{1}{w})$ is at $w = 0$. The negative sign in front of the integral arises because under the mapping $w = \frac{1}{z}$, a counterclockwise loop around $z = 0$ corresponds to a **clockwise** loop around $w = 0$. Using the definition of residues at infinity, we get that $\int_{|z|=R} f(z)dz = -2\pi i \mathrm{Res}(f; \infty)$.

7c. Since f is assumed to have finitely many singularities in $\mathbf{C}$, pick R such that $R > |z_0|$ for any singularity z_0. By part (b), $\int_{|z|=R} f(z)dz = -2\pi i \mathrm{Res}(f; \infty)$. By the ordinary residue formula, $\int_{|z|=R} f(z)dz = 2\pi i \sum_{z_0 \in \mathbf{C}} Res(f; z_0)$. Hence $\sum_{z_0 \in \hat{\mathbf{C}}} \mathrm{Res}(f; z_0) = 0$.

7d. Let C be a simple closed curve enclosing all of the singularities of f in $\mathbf{C}$. Then

$$\int_C f(z)dz - \int_\gamma f(z)dz = 2\pi i \sum \{\text{residues of } f \text{ outside } \gamma, \text{not including } \infty\}.$$

Now $\int_C f(z)dz = -2\pi i \mathrm{Res}(f;\infty)$, so

$$\int_\gamma f(z)dz = -2\pi i \sum \{\text{residues of } f \text{ outside } \gamma, \text{including } \infty\}.$$

7e. Put $P(z) = p_n z^n + p_{n-1}z^{n-1} + \cdots + p_0$ and $Q(z) = q_m z^m + q_{m-1}z^{m-1} + \cdots + q_0$. Here p_n and q_m are nonzero, and $m - n > 2$. Then

$$\begin{aligned} \frac{P(z)}{Q(z)} &= \frac{P(\frac{1}{w})}{Q(\frac{1}{w})} \\ &= \frac{p_n w^{-n} + p_{n-1}w^{-(n-1)} + \cdots + p_0}{q_m w^{-m} + q_{m-1}w^{-(m-1)} + \cdots + q_0} \\ &= \frac{p_n w^{m-n} + p_{n-1}w^{m-n+1} + \cdots + p_0 w^n}{q_m + q_{m-1}w + \cdots + q_0 w^m}. \end{aligned}$$

Hence

$$-\frac{1}{w^2}\frac{P(\frac{1}{w})}{Q(\frac{1}{w})} = -\frac{p_n w^{m-n-2} + p_{n-1}w^{m-n-1} + \cdots + p_0 w^{n-2}}{q_m + q_{m-1}w + \cdots + q_0 w^m}.$$

Since $m-n > 2$, we see that $-\frac{1}{w^2}\frac{P(\frac{1}{w})}{Q(\frac{1}{w})}$ has a removable singularity at $w = 0$. Hence the residue of $\frac{P}{Q}$ at $z = \infty$ is 0.

7f. Using parts (d) and (e), we observe that there is no residue at infinity; the only residue outside the unit circle is at the pole $z_0 = 2$. The residue there is -5^{-11}. By part (d), the integral is $(2\pi i)(5^{-11})$.