Solution Set 6

Math 113
November 26, 2001

1. a) *If* Log *denotes the principal branch of the logarithm, simplify the expressions*

$$e^{\mathrm{Log}(i)}, \mathrm{Log}(-i), i^{\mathrm{Log}(-i)}, (1+i)^{\mathrm{Log}(1+i)},$$

where $z^w = e^{w \,\mathrm{Log}\, z}$.

$e^{\mathrm{Log}(z)} = z$ for all z. For the second one, $\mathrm{Log}(-i) = \log|i| + \mathrm{Arg}(-i) = 0 + i\frac{-\pi}{2} = -i\frac{\pi}{2}$. By the same logic as for the second part $\mathrm{Log}(i) = i\frac{\pi}{2}$, and we get $i^{\mathrm{Log}(-i)} = e^{\frac{\pi^2}{4}}$. For the fourth part, we have $\mathrm{Log}(i+1) = \log(\sqrt{2}) + i\frac{\pi}{4} = \frac{\log(4)+i\pi}{4}$. So $(1+i)^{\mathrm{Log}(1+i)} = e^{\left(\frac{\log(4)+i\pi}{4}\right)^2} = 2^{\frac{i}{4}\pi}\, e^{\frac{-\pi^2}{16} + \frac{\log(2)^2}{4}}$.

b) *Find the Taylor expansion of* $\mathrm{Log}(z)$ *around* $z = 1$.

We know that $e^{\mathrm{Log}(z)} = z$ so $\mathrm{Log}'(z)z = 1$ so $\mathrm{Log}'(z) = 1/z$. Thus we get the same Taylor series that we would expect from real analysis:

$$\sum_{k=0}^{\infty} \frac{(-1)^k (z-1)^{k+1}}{k+1}$$

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2. *Show that*

$$f(z) = \int_0^1 \frac{\sin(zt)}{t} dt$$

is an entire function (a) by applying Morera's theorem, and (b) by obtaining a power series expansion for f. What is f'(z)?

The hypothesis of Morera's theorem assums the continuity of f, so we establish this. By inspection, $\frac{\sin(zt)}{t}$ is a continuous function in z and t away from $t = 0$. At $(t = 0, z)$ we define f to be z and since $lim_{t\to 0}\frac{\sin(zt)}{t} = z$ (by l'Hôpital's rule or writing out the power series), $\frac{\sin(zt)}{t}$ is continuous and thus f is. Let Γ be the boundary of an arbitrary rectangle in $\mathbb{C}$. As this rectangle is arbitrary, if we can show that the integral of f over this curve is 0 we know that f is entire.

$$\int_\Gamma f(z)dz = \int_\Gamma \int_0^1 \frac{\sin(zt)}{t}\, dt\, dz = \int_0^1 \int_\Gamma \frac{\sin(zt)}{t}\, dz\, dt$$

where we can switch the order of integration because $\frac{\sin(zt)}{t}$ is continuous by Fubini's theorem. We know by the closed curve theorem that the integral over Γ of $\frac{\sin(zt)}{t}$ is 0 , which implies

$$\int_\Gamma f(z)dz = \int_0^1 0\, dt = 0$$

as desired.

We have

$$f(z) = \int_0^1 \frac{\sin(zt)}{t}\, dt = \int_0^1 \left(\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1} t^{2k}}{(2k+1)!} \right) dt = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)(2k+1)!}$$

where we can switch the order of integration and summation by the uniform convergence of power series. Because the coefficients of this power series are dominated by those for the entire function sin, f is entire. We can then differentiate this power series to find $f'(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k+1)!} = \frac{\sin(z)}{z}$.

3.

Find Laurent series expansions for the following functions around $z_0 = 0$ Classify the type of singularity at zero, if there is one, in each of the above five examples.

a) $\sin(1/z)$

$$\sin(1/z) = \sum_{k=0}^{-\infty} \frac{(-1)^k z^{2k-1}}{2|k| + 1!}.$$

Since sin is entire and $1/z$ is analytic away from 0, this Laurent series is valid for $z \neq 0$. Since infinitely many of our coefficients are zero, we have an essential singularity at $z_0 = 0$.

b) $z/(z+1)$

We know that $1/(z+1) = 1/(1-(-z))$ which for $|z| < 1$ is a convergent geometric series. Multiplying this by z we get:

$$z/(z+1) = \sum_{k=0}^{\infty} = (-1)^k z^{k+1}$$

Which we know converges in the annulus given by $R_1 = 0$ and $R_2 = 1$. From this it's clear that we don't have a singularity of any type at $z_0 = 0$.

c) e^z/z^2

We know that $e^z = \sum z^k/k!$, so we can just divide this by z^2 and re-index to get:

$$e^z/z^2 = \sum_{k=-2}^{\infty} z^k/(k+2)!$$

Because for $k > 2$ $a_{-k} = 0$, $R_1 = 0$; by the ratio test

$R_2 = \infty$ as we have the reciprocal of something that goes to zero. This function has a pole of order 2 at $z_0 = 0$, as a_{-2} is the non-zero coefficient with the lowest subscript.

d) $1/(e^z - 1)$ *(Find the first three terms only).*

We know that $e^z = \sum z^k/k!$, so we can just divide this by z^2 and re-index to get:

Using long division and the Taylor series for $e^z - 1$, the first three terms are:

$$\frac{1}{z} - \frac{1}{2} + \frac{z}{12}$$

Because $1/(e^z - 1)$ is analytic except at $z = 0$, the Laurent series is valid except at $z = 0$.

e) $\frac{(z-1)^2(z+3)}{1-\sin(\pi z/2)}$ *(Find the first three terms only).*

By long division, we get the first three terms as:

$$3 + \left(-5 + \frac{3\,\pi}{2}\right)\, z + \left(1 - \frac{5\,\pi}{2} + \frac{3\,\pi^2}{4}\right)\, z^2$$

At zero this function take the value 3, so there is no singularity there. Since our series above

is just a Taylor series, and since this has a singularity at $z = 1$, we have $R_1 = 0$, $R_2 = 1$.

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4. *Prove the following complex version of l'Hôpital's rule: Let $f(z), g(z)$ be analytic, both having seros of order k at z_0. Then $f(z)/g(z)$ has a removable singularity at z_0, and*

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f^{(k)}(z_0)}{g^{(k)}(z_0)}.$$

We know that $f(z) = (z - z_0)^k f_1(z)$ and $g(z) = (z - z_0)^k g_1(z)$ with $f_1(z)$ and $g_1(z)$

non-zero z_0. We know there is a singularity for $f(z)/g(z)$ at z_0 as $g(z_0) = 0$. Also $lim_{z \to z_0}(z - z_0)f(z)/g(z) = lim_{z \to z_0}(z - z_0)f_1(z)/g_1(z)$, which implies that $lim_{z \to z_0}(z - z_0)f(z)/g(z) = 0$ and the other part of the limit is defined so the total limit is 0. Thus by Riemann's

removable singularity principle we know that the singularity at z_0 is removable.

For the second part, we know that $f(z) = (z - z_0)^k f_1(z)$. From this we know that

$f'(z) = k(z - z_0)^{k-1} f_1(z) + (z - z_0)^k f_1'(z)$. When we continue taking derivatives, by applying

the product rule we see that $f^{(k)}(z) = k!(z - z_0)^0 f_1(z) + \ldots + (z - z_0)^k f_1^{(k)}(z)$ where all the middle terms have some factor of $(z - z_0)$. Thus $f^{(k)}(z_0) = k!f_1(z_0)$. By the same logic $g^{(k)}(z_0) = k!g_1(z_0)$, thus:

$$\frac{f^{(k)}(z_0)}{g^{(k)}(z_0)} = \frac{f_1(z_0)}{g_1(z_0)} = \lim_{z\to z_0} \frac{f(z)}{g(z)}$$

the

last step resulting from our definition of f_1, g_1.

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5. *We say that a function $f(z)$ has an isolated singularity at infinity iff $g(w) = f(1/w)$ has an isolated singularity at $w = 0$.*

a) *Show that f has an isolated singularity at infinity iff f is analytic outside of some bounded subset of $\mathbf{C}$. Sketch a picture of what this means in terms of the Riemann sphere.*

If f is analytic outside of some bounded subset of $\mathbf{C}$, there exists an r such that f is analytic on $\{z \in \mathbf{C} : |z| > r\}$. By the chain rule, $f(1/w)$ is analytic on $\{z \in \mathbf{C} : 0 < |z| < 1/r\}$. So f has an isolated singularity at infinity.

If f has an isolated singularity at infinity, there is a deleted circular neighborhood of 0, say of radius r, on which $f(1/w)$ is analytic. By the chain rule, $f(1(1/z)) = f(z)$ is analytic on $\{z \in \mathbf{C} : |z| > 1/r\}$, so f is analytic outside of the bounded subset of $\mathbf{C}$ $\{z \in \mathbf{C} : |z| < 1/r\}$.

b) *Show that a polynomial of defree $N \geq 1$ has a pole of order N at infinity.*

We define having a pole of order N at infinity in the logical

manner given the definition of an isolated singularity. Let $P(z)$ be a polynomial of degree N,

with $N \geq 1$. Let $g(w) = P(1/w)$. Let $P(z) = a_0 + a_1 z + \ldots + a_N z^N$; because P has degree

N, $a_N \neq 0$. Then

$$g(w) = a_0 + a_1/w + \ldots + a_N/w^N = \frac{a_0 w^N + a_1 w^{N-1} + \ldots + a_N}{w^N}$$

Let $A(w) = a_0 w^N + a_1 w^{N-1} + \ldots + a_N$, then $A(0) = a_N \neq 0$. Let $B(w) = w^N$. Then the order

of zero of B is N, so g has a pole of order N at 0, and so f has a pole of order N at infinity

c) *Show that an entire function which is not a polynomial must have an essential singularity at infinity.*

If f is an entire function we know that f can be expressed as a power series about 0, i.e.

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \text{ for some constants } \{a_k\}$$

Because f is not a polynomial we know this power series is infinite, i.e. for any $N > 0$ with

$a_N \neq 0$ there is some $n > N$ s.t. $a_n \neq 0$, as otherwise f would a polynomial of degree less than

N. Now consider:

$$g(w) = f(1/w) = \sum_{k=0}^{\infty} a_k z^{-k} = \sum_{j=-\infty}^{0} b_j z^j$$

where $b_j = a_{-j}$ for $j \leq 0$ and $b_j = 0$ for $j > 0$. This is a Laurent series, and we know from

their definitions that $R_1 = 0$ as all the terms to the right are zero, and $R_2 = \infty$, as the a_k are coefficients for an everywhere convergent power

series. Because $b_j \neq 0$ for infinitely many $j < 0$ (because f is not a polynomial again), we

know from class that g has an essential singularity at 0, so f has an essential singularity at

infinity.

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6. *Suppose $f(z)$ is analytic in the annulus $R_1 < |z| < R_2$, and let r be such that $R_1 < r < R_2$.*

a) *Show that f can be developed on the circle of radius r around $z = 0$ as a Fourier series:*

$$f(re^{i\theta}) = \sum_{n=-\infty}^{\infty} b_n e^{in\theta}.$$

We know f can be expressed as a Laurent series on the annulus $R_1 < |z| < R_2$, $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$. Letting $z = re^{i\theta}$, we can let $b_n = r^n a_n$ and we have $f(re^{i\theta}) = \sum_{n=-\infty}^{\infty} b_n e^{in\theta}$.

b) *Show that the Fourier coefficients b_n can be calculated directly from f using the Fourier inversion formula:*

$$b_n = \frac{1}{2} \int_0^{2\pi} f(re^{i\theta}) e^{-in\theta} d\theta.$$

If we choose C_r the circle of radius r centered at 0, we have that that:

$$b_n = \frac{r^n}{2\pi i}\int_{C_r} \frac{f(w)}{w^{n+1}}\,dw$$

We parameterize with $z(\theta) = re^{i\theta}$ with $\theta \in [0, 2\pi]$, noticing that $\dot{z}(\theta) = rie^{i\theta}$. Thus we have:

$$b_n = \frac{r^n}{2\pi i}\int_0^{2\pi} \frac{f(re^{i\theta})}{r^{n+1}e^{i(n+1)\theta}} rie^{i\theta}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})e^{-in\theta}\,d\theta.$$

c) *Prove Parseval's formula:*

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=-\infty}^{\infty} |b_n|^2.$$

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})\overline{f(re^{i\theta})} d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta}) \sum_{n=-\infty}^{\infty} \overline{b_n} e^{-in\theta} d\theta$$

by conjugating both sides of part (a). We may switch the order of summation and integration because power series converge uniformly, so we have

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \overline{b_n}\int_0^{2\pi} f(re^{i\theta})e^{-in\theta} d\theta = \sum_{n=-\infty}^{\infty} \overline{b_n} b_n$$

by part (b), and we are done.

These solutions were adapted from Arthur Rudolph's set, with TeX help from Joe Rabinoff.