Problem Set 2

October 11, 2001

1. Find all complex solutions of $e^z = -5$:

$e^{x+iy} = e^x(\text{cos}y + i\text{sin}y) = 5(\text{cos}\pi + i\text{sin}\pi)$. Note: if $e^{r_1}\text{cis}\theta_1 = e^{r_2}\text{cis}\theta_2$ then $r_1 = r_2$ and $\theta_1 - \theta_2 = 2\pi k$ for $k \in \mathbb{Z}$. (Pf: $|e^{r_i}\text{cis}\theta_i| = e^{r_i} \Rightarrow r_1 = r_2$. This gives $\text{cis}\theta_1 = \text{cis}\theta_2$. Since Cos is 1-1 in each quadrant and comparing the signs of Sin and Cos determine the quadrant of an angle, this implies $\theta_1 = \theta_2$.) So $\mathbf{z} = \ln\mathbf{5} + (\mathbf{2n}+\mathbf{1})\pi\mathbf{i}$.

2. Find all complex solutions of $\sin(z) = 5$

$$\begin{aligned}
\text{sin}z &= \frac{e^{iz} + e^{-iz}}{2i} = 5 \\
&\Rightarrow (e^{iz})^2 - 10ie^{iz} - 1 = 0 \\
\Rightarrow e^{iz} &= \frac{10i \pm \sqrt{(100i)^2 + 4}}{2} = i(5 \pm \sqrt{24}) \\
&\Rightarrow iz = (\frac{1}{2} + 2n)\pi i + \ln(5 \pm \sqrt{24}) \\
\mathbf{z} &= (\frac{\mathbf{1}}{\mathbf{2}} + \mathbf{2n})\pi - \mathbf{i}\ln(\mathbf{5} \pm \sqrt{\mathbf{24}})
\end{aligned}$$

Note: The solutions are closed under complex conjugation as $\ln(5+\sqrt{24}) = -\ln(5-\sqrt{24})$ (Pf: $\ln(5+\sqrt{24}) + \ln(5-\sqrt{24}) = \ln((5+\sqrt{24})(5-\sqrt{24}) = \ln 1 = 0$. Q: Why must this be the case? A: When we estend *real* functions f to the complex plane $\bar{f(z)} = f(\bar{z})$ because all the coefficients of the power series of f are real.

3. If g is an analytic function on the open set $A \subseteq C$ and $B := \{z \in A | g(z) \neq 0\}$, show that B is open and that $1/g$ is analytic on B.

g analytic on A implies that g is continuous there because $\lim_{h\to 0}(g(z+h) - g(z)) = \lim_{h\to 0}(h \cdot \frac{g(z+h)-g(z)}{h}) = 0 * g'(z) = 0$. The ϵ - δ definition of continuity is equivalent to the statement that the inverse image of open sets is open. Since $\mathbb{C} - 0$ is open, $g^{-1}(\mathbb{C} - 0) = B$ is open.

Alternatively, $z \in B$ means $|g(z)| = h > 0$. Since g is continuous there exists $\delta > 0$ such that $|z' - z| < \delta$ implies $|g(z') - g(z)| < \frac{h}{2}$. $|g(z) - g(z')| + |g(z')| \geq |g(z)|$ by the triangle inequality. This implies $|g(z')| \geq h - \frac{h}{2} = \frac{h}{2}$. In other words the ball around z of radius δ is in B and thus B is open.

To show $1/g$ is analytic on B:

$$\lim_{h\to 0}\left(\frac{1}{h}\left(\frac{1}{g(z+h)} - \frac{1}{g(z)}\right)\right) = \lim_{h\to 0}\left(\frac{\frac{1}{g(z+h)} - \frac{1}{g(z)}}{h} \cdot \frac{1}{g(z+h)g(z)}\right).$$

Since g is continuous and $g(z) \neq 0$, $\lim_{h\to 0}\frac{1}{g(z+h)g(z)}$ exists and is equal to $\frac{1}{g(z)^2}$. (Pf: fix $\epsilon > 0$. $|\frac{1}{g(z+h)g(z)} - \frac{1}{g(z)^2}| = |\frac{g(z)-g(z+h)}{g(z+h)g(z)}|$. Let $r = |g(z)|$ and take δ such that $|g(z) - g(z+h)| < \frac{r^3}{2}\epsilon$ and $|g(z) - g(z+h)| < \frac{r}{2}$ whenever $|h| < \delta$. $|\frac{g(z)-g(z+h)}{g(z+h)g(z)}| < \frac{\epsilon r^3/2}{(r-r/2)r^2} = \epsilon$, whenever $|h| < \delta$.

$\lim_{h\to 0}\frac{g(z)-g(z+h)}{h} = -g'(z)$ by analyticity of g. This implies $\lim_{h\to 0}(\frac{1}{h}(\frac{1}{g(z+h)} - \frac{1}{g(z)}))$ exists everywhere in B, so $1/g$ is analytic on B (since B is a nbhd of every point in B).

4. There didn't seem to be much trouble with this.

5. Show that if $f, g : \mathbb{C} \to \mathbb{C}$ are entire, then $g \circ f$ is entire and

$$(g \circ f)'(z) = g'(f(z)) \cdot f'(z).$$

Fix $z_0 \in \mathbb{C}$. Let $\epsilon(h) = \frac{1}{h}(g(z_0 + h) - g(z_0) - hg'(z_0))$. This can be rewritten

$$(g'(z_0) + \epsilon) \cdot h = g(z_0 + h) - g(z_0).(*)$$

g differentiable at z_0 is the condition that $\lim_{h\to 0}\epsilon = 0$ (Pf: g differentiable at $z_0 \Leftrightarrow \lim_{h\to 0}\frac{g(z_0+h)-g(z_0)}{h}) = g'(z_0) \Leftrightarrow \lim_{h\to 0}\frac{g(z_0+h)-g(z_0)-hg'(z_0)}{h}) = 0 \Leftrightarrow \lim_{h\to 0}\epsilon = 0$.)

For $z_0 = f(w_0)$ and $h = f(w_0 + w) - f(w_0)$ (*) reads

$$(g'(z_0) + \epsilon)(f(w_0 + w) - f(w_0)) = g(f(w_0 + w)) - g(f(w_0))(**).$$

We claim $\lim_{w\to 0}\epsilon(f(w_0+w) - f(w_0)) = \lim_{h\to 0}\epsilon(h) = 0$. (Pf: This follows from continuity of f. Specifically fix $\epsilon' > 0$. Because $\lim_{h\to 0}\epsilon(h) = 0$, there exists a δ such that $|h| < \delta$ implies $|\epsilon(h)| < \epsilon'$. By continuity of f, there exists a δ' such that $|w| < \delta'$ implies $|f(w_0 + w) - f(w_0)| < \delta \Rightarrow |\epsilon(f(w_0 + w) - f(w_0))| < \epsilon'$.) Dividing both sides of (**) by w and taking the limit as $w \to 0$, gives the desired result.

Note: $g'(f(z))$ is defined to be $\lim_{h\to 0}\frac{g(f(z)+h)-g(f(z))}{h})$, but it also equals $\lim_{h'\to 0}\frac{g(f(z+h'))-g(f(z))}{f(z+h')-f(z)})$ where it exists by continuity of f. (Pf: Fix $\epsilon > 0$. g differentiable means there exists a δ such that $|\frac{g(f(z)+h)-g(f(z))}{h}) - g'(f(z))| < \epsilon$ whenever $|h| < \delta$. This implies

$|\frac{g(f(z+h))-g(f(z))}{f(z+h)-f(z)}) - g'(f(z))| < \epsilon$ whenever $|f(z+h) - f(z)| < \delta$.) Namely it doesn't matter how fast you approach 0, excpet you must do it at the same pace in the denominator and the difference between the values you're acting g on in the numerator.

Last note: These proofs are the same as in real analysis because we have only used the properties of a nomed vector space.