Mathematics 113 Fall 2001

Study sheet for final exam

The final exam will cover everything in Chapters 1-11 and 13 of the course text, excluding a few results we skipped, e.g. the Schwarz Reflection Principle. I will of course emphasize material which was specifically covered in lectures or in the homework. In certain cases (e.g. simple connectedness) we defined things differently in class than the way they are defined in the book. You should follow the class notes in such cases.

The exam will emphasize definitions, statements, and applications of the main results of the class. It will contain some proofs but will not be excessively proof-oriented. For example, you will not be asked to prove any difficult results (such as the Cauchy Integral Theorem or Residue Theorem) from scratch. I am mainly interested in seeing if you know how to use such results.

You should know the statements of the following results for the final exam. (This is not an exhaustive list, just a sample).

1. Uniqueness theorem for power series / analytic functions
2. Cauchy integral formula (and its generalizations, e.g. Theorem 10.11)
3. Liouville's theorem
4. Mean value principle
5. Maximum modulus principle
6. Morera's theorem
7. Casorati-Weierstrass theorem
8. Riemann's removable singularity theorem
9. Cauchy's residue theorem
10. The argument principle
11. Rouché's theorem

12. The Schwarz lemma

13. Classification of automorphisms of the unit disk

14. Existence and uniqueness of bilinear mappings (Theorem 13.23)

Sample final exam questions

1. Compute all complex fourth roots of $-i$.

2. Find all complex solutions of $\cos(z) = 2$.

3. Prove that if $f(z)$ and $\overline{f(z)}$ are both analytic in a domain D, then f is constant throughout D.

4. Evaluate the line integral $\int_C y\,dz$, where C is the line segment from 0 to $2 + 3i$.

5. Prove that
$$|\int_C \sin z^2\,dz| \le e\sqrt{2}$$
where C is the line segment in $\mathbf{C}$ from 0 to $1 + i$.

6. Find the power series expansion for e^z about any point $a \in \mathbf{C}$.

7. Find a closed form for the Laurent expansion of $f(z) = \frac{1}{(z-2)(z-3)^2}$ around $z = 2$. For which values of z does this Laurent series converge?

8. Define the sequence $\{a_n\}$ by requiring that
$$1 - x^2 + x^4 - x^6 + \cdots = \sum_{n=0}^{\infty} a_n (x-3)^n$$
for all real $0 \le x \le 1$. What is
$$\limsup_{n\to\infty} \left(|a_n|^{1/n}\right)?$$

9. Determine and classify all singularities of $\frac{z\cot(\pi z/2)}{z+1}$, and find the residues at those singularities.

10. Evaluate
$$\int_{|z|=1} \sin\frac{1}{z} dz.$$

11. Evaluate the integral
$$\int_0^{2\pi} \frac{d\theta}{5 - 4\cos\theta}.$$

12. Evaluate
$$\int_{-\infty}^{\infty} \frac{dx}{(1 + x + x^2)^2}.$$

13. Fix an integer $n \geq 1$ and a real number $r > 0$. What is the maximum modulus of the function $z^n + i$ on the disk $|z| \leq r$, and where is this maximum modulus achieved?

14. Prove or disprove: There exists an analytic mapping from $\mathbf{C}$ to the upper half-plane $\mathrm{Im}(z) > 0$.

15. Find the number of zeros of $f(z) = z^6 - 15z^2 + 1$ in the annulus $1 \leq |z| \leq 2$.

16. Suppose f is an analytic function which maps the unit disc $|z| < 1$ to itself. If $f(1/2) = 0$, what is the largest possible value for $|f(7/8)|$?

17. Find a bilinear mapping sending $0, 1, 2$ to $1, 0, \infty$, respectively.

18. Find a conformal isomorphism from the left half-plane $\mathrm{Re}(z) < 0$ to the open unit disk $|z| < 1$.

19. Suppose f is analytic on the disk $|z| < 2$ except at $z = 1$, where f has a simple pole. If
$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$
is the Taylor series expansion for f around $z = 0$, prove that $\lim_{n\to\infty} a_n$ exists and equals the residue of f at $z = 1$.

20. Prove that the image of $\mathbf{C}$ under a nonconstant entire mapping is dense in $\mathbf{C}$.