Homework Assignment # 5

Due Thursday, October 25

1. Find power series expansions for $f(z) = \frac{1}{(z+1)^2}$, $g(z) = e^z$, and $h(z) = z^3$ centered at $z = 1+i$. What are the radii of convergence of these power series?

2. a. Find all complex zeros of the functions $\sin(z)$ and $\cos(z)$.

   b. Suppose f is an analytic function on the open set $U = \mathbf{C} \backslash \{\frac{\pi}{2} \pm \pi k \mid k \in \mathbf{Z}\}$, and that $f(\frac{\pi}{n}) = \tan(\frac{\pi}{n})$ for all integers $n \geq 3$. Show that f is not entire, and that f does not attain the value i.

3. Define the *Bernoulli numbers* B_{2n} by the formula

$$\frac{z}{2}\cot(\frac{z}{2}) = 1 - B_2\frac{z^2}{2!} - B_4\frac{z^4}{4!} - B_6\frac{z^6}{6!} - \cdots$$

   (This sequence of rational numbers is very important in number theory, and is intimitely connected with both the distribution of prime numbers and Fermat's Last Theorem.)

   a. Explain why there are no odd terms in the above power series. [**Hint:** Prove that this is a general property of odd functions.]

   b. Compute the Bernoulli numbers B_2, B_4, B_6, B_8.

   c. What is the radius of convergence of the above power series?

4. a. Show directly that the maximim and minimum moduli of e^z on a compact domain D are assumed on the boundary of D.

   b. Use the Minimum Modulus Principle to give another proof of the Fundamental Theorem of Algebra.

5. a. Let $D \subseteq \mathbf{C}$ be a domain, and let $E \subseteq D$ be a subset containing an accumulation point. Let $F(z,w) : D \times D \to \mathbf{C}$ be a function which is analytic in z for each fixed $w \in D$, and analytic in w for each fixed $z \in D$. If $F(z,w) = 0$ whenever $z, w \in E$, show that $F(z,w) = 0$ for all $z, w \in D$.

b. Use part (a) to show that $e^{z+w} = e^z \cdot e^w$ and $\cos(z+w) = \cos(z)\cos(w) - \sin(z)\sin(w)$ for all $z, w \in \mathbf{C}$, assuming the corresponding identities for $z, w \in \mathbf{R}$. [Use the power series definitions for $e^z, \sin(z), \cos(z)$ as was done in class.]

6. a. Prove the following generalized Cauchy integral formula: If f is entire, C is a circle of radius R centered at z_0, and if a is inside C, then

$$f^{(k)}(a) = \frac{k!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{k+1}} dz.$$

[**Hint:** First find complex numbers $c_0, \ldots, c_k \in \mathbf{C}$ such that

$$\int_C \frac{f(z)}{(z-a)^{k+1}} dz = c_0 \int_C \frac{dz}{(z-a)^{k+1}} + c_1 \int_C \frac{dz}{(z-a)^k} + \cdots + c_k \int_C \frac{dz}{(z-a)} + \int_C g(z) dz$$

with g entire.]

b. Use part (a) to evaluate

$$\int_{|z|=1} \frac{e^{2z}}{z^m} dz$$

for all $m \in \mathbf{Z}$.

c. Let $z_0 \in \mathbf{C}$, and suppose $f(z)$ is entire. If $|f(z)| \leq M$ whenever $|z - z_0| = R$, show that

$$|f^{(k)}(z_0)| \leq \frac{k!}{R^k} M$$

for all $k \geq 0$. Use this inequality with $k = 1$ to give another proof of Liouville's theorem.