Midterm, Mathematics 101

Solutions

Problem 1. Give an example of set A such that $\cap A = \emptyset$ but $\cap B \neq \emptyset$ for every finite nonempty subset $B \subset A$
Solution. Let $A = \{C_0, C_1, \dots\}$, where $C_n = \{i \in \mathbb{N} : i \neq n\}$. Then, since $n \notin C_n$, $\cap A = \emptyset$. On the other hand, if B is a finite subset of A, and $m = \max\{n : C_n \in B\} + 1$, then $m \in \cap B$, so $\cap B \neq \emptyset$
Problem 2. Give an example of an injective map $f : \mathbb{R} \to \mathbb{R}$ which is not surjective.
Solution. You can take $f(x) = e^x$.
Problem 3. Prove that if $|A| = |B|$ then $|P(A)| = |P(B)|$.
Solution. Let $f : A \to B$ be a bijection. Define $F : P(A) \to P(B)$ by $F(C) = \{f(c), c \in C\}$ for every $C \in P(A)$.
If $F(C_1) = F(C_2)$, then $f(C_1) = f(C_2)$, so, since f is an injection, $C_1 = f^{-1}(f(C_1)) = f^{-1}(f(C_2)) = C_2$. Thus F is injective.
Now pick any $D \in P(B)$. Take $C = f^{-1}(D)$. Since f is surjective, $f(C) = D$. Thus F is surjective.
Problem 4. Let G be a group for which $g^2 = 1$ for all $g \in G$. Prove that G is abelian.
Solution. Let $g_1, g_2 \in G$. Then $(g_1 * g_2)^2 = 1$. Multiply this equality by g_2 from the right and by g_1 from the left and use assciativity and the fact that $g_1^2 = g_2^2 = 1$. Then we get $g_1 * g_2 = g_2 * g_1$.
Problem 5. Give an example of a group G with subgroups H and K such that $H \cup K$ is not a subgroup of G.
Solution. Take $G = (\mathbb{C}, +), H = (\mathbb{R}, +), K = (i\mathbb{R}, +)$. Then H and K are subgroups of G, but $H \cup K$ is not closed under addition ($1 \in H \subset H \cup K$, $i \in K \subset H \cup K$, but $1 + i \notin H \cup K$).