Poincare Spaces and Spivak Fibrations (Lecture 26)

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Let X be a topological space and $\mathcal{C}$ an ∞-category. We let $\mathrm{Shv}_{\mathrm{lc}}(X;\mathcal{C})$ denote the ∞-category of maps from the Kan complex $\mathrm{Sing}_\bullet(X)$ into $\mathcal{C}$. We will refer to $\mathrm{Shv}_{\mathrm{lc}}(X;\mathcal{C})$ as the ∞-category of *locally constant* $\mathcal{C}$-valued sheaves on X, or sometimes as the ∞-category of *local systems* of $\mathcal{C}$-valued sheaves on X. If X is a polyhedron with triangulation T, then we can identify $\mathrm{Shv}_{\mathrm{lc}}(X;\mathcal{C})$ with the full subcategory of $\mathrm{Shv}_T(X;\mathcal{C})$ spanned by those functors which carry each inclusion $\tau \subseteq \tau'$ of simplices to an invertible morphism of $\mathcal{C}$.

Let $f : X \to Y$ be a map of topological spaces. Then f induces a pullback functor $f^* : \mathrm{Shv}_{\mathrm{lc}}(Y;\mathcal{C}) \to \mathrm{Shv}_{\mathrm{lc}}(X;\mathcal{C})$. Suppose that $\mathcal{C}$ is the ∞-category of spectra. Then f^* preserves all limits and colimits, and therefore admits both a left adjoint $f_!$ and a right adjoint f_*.

In the special case where Y is a point, we will denote the functors $f_!$ and f_* by $C_*(X;\bullet)$ and $C^*(X;\bullet)$, respectively. If X si a polyhedron with triangulation T, these are described by the formulas

$$C_*(X;\mathcal{F}) = \varinjlim_{\tau} \mathcal{F}(\tau) \qquad C^*(X;\mathcal{F}) = \varprojlim_{\tau} \mathcal{F}(\tau).$$

If X is a *finite* polyhedron, we conclude that the construction $C^* : \mathrm{Shv}_{\mathrm{lc}}(X;\mathrm{Sp}) \to \mathrm{Sp}$ commutes with homotopy colimits.

Suppose now that X is connected with base point x. Then $\mathrm{Shv}_{\mathrm{lc}}(X;\mathrm{Sp})$ can be identified with the ∞-category of modules over the A_∞-ring $R = \Sigma^\infty_+ \Omega(X)$. Any functor $F : \mathrm{LMod}_R \to \mathrm{Sp}$ is determined by its value $F(R) \in \mathrm{Sp}$, together with its right R-module structure. Indeed, the fact that F commutes with homotopy colimits implies that F is given by $F(M) \simeq F(R) \wedge_R M$. We can identify $F(R)$ with a local system ζ on X, so that F is given by the formula $F(M) = C_*(X; M \wedge \zeta)$. This description generalizes immediately to the case where X is not assumed to be connected:

Proposition 1. *Let $F : \mathrm{Shv}_{\mathrm{lc}}(X;\mathrm{Sp}) \to \mathrm{Sp}$ be a functor which commutes with homotopy colimits. Then F is given by $F(\mathcal{F}) = C_*(\mathcal{F} \wedge \zeta)$, where ζ is a local system of spectra on X. Moreover, the local system ζ is determined uniquely up to equivalence.*

In particular, if X is a finite polyhedron (or any space equivalent to a finite polyhedron) and $f : X \to *$ denotes the projection map, we have an equivalence of functors

$$f_*(\bullet) \simeq C_*(\bullet \wedge \zeta_X)$$

for some local system ζ_X on X.

Definition 2. We say that a finite polyhedron X is a *Poincare space* if ζ_X is a spherical fibration (that is, if each of the fibers $\zeta_X(x)$ is an invertible spectrum). In this case, we say that ζ_X is the *Spivak normal fibration* of X.

Remark 3. Let X be a finite polyhedron containing a point x. Let $i : \{x\} \to X$ denote the inclusion and $p : X \to *$ the projection map, so that $p \circ i$ is a homeomorphism. Then we have a homotopy equivalence of spectra

$$i^*\zeta_X \simeq (p \circ i)_! i^* \zeta_X \simeq p_!(i_! i^* \zeta_X) \simeq p_!((i_! S) \wedge \zeta_X) \simeq C^*(X; i_! S).$$

In other words, the stalk $\zeta_X(x)$ is given by taking global sections of the local system of spectra on X that assigns to each point $y \in X$ the suspension spectra $\Sigma^{\infty}_{+} P_{x,y}$, where $P_{x,y}$ denotes the path space $\{p : [0,1] \to X : p(0) = x, p(1) = y\}$.

Remark 4. Let $\underline{S}$ denote the constant local system on X with value the sphere spectrum, so we have a canonical map $S \to C^*(X; \underline{S}) \simeq C_*(X; \underline{S} \wedge \zeta_X) \simeq C_*(X; \zeta_X)$. We can identify this map with a point of $\Omega^{\infty} C_*(X; \zeta_X)$, which we will refer to as the *fundamental class* of X and denote by $[X]$.

The fundamental class determines the equivalence of functors $C^*(X; \bullet) \simeq C_*(X; \bullet \wedge \zeta_X)$: it is given by

$$C^*(X; \mathcal{F}) \simeq \mathrm{Mor}(\underline{S}, \mathcal{F}) \to \mathrm{Mor}(\zeta_X, \mathcal{F} \wedge \zeta_X) \to \mathrm{Mor}(C_*(X; \zeta_X), C_*(X; \mathcal{F} \wedge \zeta_X)) \stackrel{[X]}{\to} C_*(X; \mathcal{F} \wedge \zeta_X).$$

Example 5. Let X be a simply connected finite polyhedron. Then X is a Poincare space if and only if there exists a fundamental class $\eta_X \in \mathrm{H}_n(X; \mathbf{Z})$ which induces cap product isomorphisms $\phi_i : \mathrm{H}^i(X; \mathbf{Z}) \to \mathrm{H}_{n-i}(X; \mathbf{Z})$. The "only if" direction is obvious: if X is a Poincare space, then $\zeta_X \wedge \mathbf{Z}$ is necessarily equivalent to $\Sigma^{-n} \underline{\mathbf{Z}}$ (orientability is obvious, since X is simply connected) so we can take η_X to be the image of the fundamental class $[X]$; the desired result then follows from the equivalence

$$C^*(X; \underline{\mathbf{Z}}) \simeq C_*(X; \underline{\mathbf{Z}} \wedge \zeta_X) \simeq C_*(X; \Sigma^{-n} \underline{\mathbf{Z}}).$$

The converse requires the simple connectivity of X. Note that η_X induces a map of spectra $C^*(X; \underline{\mathbf{Z}}) \to \Sigma^{-n} \mathbf{Z}$, hence a map $C_*(X; \underline{Z} \wedge \zeta_X) \to \Sigma^{-n} \mathbf{Z}$ which is adjoint to a map $\theta : \underline{Z} \wedge \zeta_X \to \Sigma^{-n} \underline{\mathbf{Z}}$. We claim that θ is invertible (from which it will follow that each fiber of ζ_X is equivalent to the invertible spectrum $\Sigma^{-n} S$). Since X is simply connected (and the fibers of ζ_X are k-connective for $k \ll 0$), it will suffice to show that θ induces an equivalence after applying the functor C_*. That is, we must show that the canonical map

$$C^*(X; \underline{\mathbf{Z}}) \simeq C_*(X; \underline{Z} \wedge \zeta_X) \to \Sigma^{-n} C_*(\underline{\mathbf{Z}})$$

is a homotopy equivalence. On the level of homotopy groups, this is precisely the condition that the maps ϕ_i are isomorphisms.

Let us now depart from our previous convention and regard quadratic functors as *covariant* functors from a stable ∞-category $\mathcal{C}$ to spectra. If R is an A_∞-ring with involution, we have a quadratic functor $Q^s : \mathrm{RMod}_R \to \mathrm{Sp}$ given by

$$Q^s(M) = (M \wedge_R M)^{h\Sigma_2},$$

which restricts to a nondegenerate quadratic functor on $\mathrm{RMod}_R^{\mathrm{fp}}$. If X is a space equipped with a spherical fibration ζ and $f : X \to *$ denotes the projection map, then we obtain a quadratic functor $Q_\zeta : \mathrm{Shv}_{\mathrm{lc}}(X; \mathrm{RMod}_R) \to \mathrm{Sp}$ given by the formula $Q_\zeta(\mathcal{F}) = C_*(X; \zeta \wedge Q^s(\mathcal{F}))$, which is nondegenerate when restricted to the ∞-category of compact objects of $\mathrm{Shv}_{\mathrm{lc}}(X; \mathrm{RMod}_R)$.

Let $\underline{R}$ denote the constant sheaf on X having the value R. Given a map of spectra $\eta : S \to C_*(X; \zeta)$, we obtain a map $S \to C_*(X; \zeta \wedge R^{\mathrm{h}\Sigma_2}) \simeq Q_\zeta(\underline{R})$, which we will denote by q. Then the pair $(\underline{R}, q)$ is a quadratic object of $\mathrm{Shv}_{\mathrm{lc}}(X; \mathrm{RMod}_R)$. Let B_ζ denote the polarization of Q_ζ, given by the formula

$$B_\zeta(\mathcal{F}, \mathcal{F}') = C_*(X; \mathcal{F} \wedge_R \mathcal{F}' \wedge \zeta).$$

If X is a Poincare space and $\zeta = \zeta_X$ is its Spivak normal fibration, then we have a homotopy equivalence

$$B_\zeta(\underline{R}, \mathcal{F}) = C_*(X; \underline{R} \wedge_R \mathcal{F} \wedge \zeta) \simeq C_*(X; \mathcal{F} \wedge \zeta) \simeq C^*(X; \mathcal{F}) \simeq \mathrm{Mor}(\underline{R}, \mathcal{F}).$$

This tells us that $\underline{R}$ is self-dual: that is, $(\underline{R}, q)$ is a Poincare object of $\mathrm{Shv}_{\mathrm{lc}}(X; \mathrm{RMod}_R)$. We therefore obtain an element $\sigma_X^{vs} \in \Omega^{\infty} \mathbb{L}^{vs}(X, \zeta_X, R)$, called the *visible symmetric signature* of the Poincare complex X.

For later use, we will need a slight generalization of the notion of a Poincare complex. Suppose we are given a map of finite spaces $\partial X \to X$ (which, up to homotopy equivalence, we may as well suppose is given by an inclusion between finite polyhedra). Given a local system of spectra $\mathcal{F}$ on X, we let $\mathcal{F} \mid \partial X$ denote the pullback of $\mathcal{F}$ to ∂X, and form fiber sequences

$$C_*(\partial X; \mathcal{F} \mid \partial X) \to C_*(X; \mathcal{F}) \to C_*(X, \partial X; \mathcal{F})$$

$$C^*(X, \partial X; \mathcal{F}) \to C^*(X; \mathcal{F}) \to C^*(\partial X; \mathcal{F} \mid \partial X).$$

Arguing as above, we see that $C^*(X, \partial X; \bullet)$ commutes with homotopy colimits and is therefore given by $\mathcal{F} \mapsto C_*(X; \zeta_{(X,\partial X)} \wedge \mathcal{F})$ for some local system $\zeta_{(X,\partial X)}$. The equivalence between $C^*(X, \partial X; \bullet)$ is determined by a fundamental class $[X] : S \to C_*(X, \partial X; \zeta_{(X,\partial X)})$. Note that $[X]$ determines a composite map

$$[\partial X] : S \xrightarrow{[X]} C_*(X, \partial X; \zeta_{(X,\partial X)}) \to \Sigma C_*(\partial X; \zeta_{(X,\partial X)} \mid \partial X) \simeq C_*(\partial X; \Sigma \zeta_{(X,\partial X)} \mid \partial X).$$

Definition 6. A pair of finite spaces $(X, \partial X)$ is a *Poincare pair* if the following conditions are satisfied:

(1) The local system $\zeta_{(X,\partial X)}$ defined above is a spherical fibration.

(2) The map $[\partial X]$ is a fundamental class for ∂X: that is, it induces a homotopy equivalence

$$C^*(\partial X; \mathcal{F}) \to C_*(\partial X; (\Sigma \zeta_{X,\partial X} \mid \partial X) \wedge \mathcal{F})$$

for every local system $\mathcal{F}$ on ∂X. (So that the Spivak normal fibration of ∂X is given by $\Sigma(\zeta_{X,\partial X} \mid \partial X)$.)

Remark 7. Let $(X, \partial X)$ be a pair of finite spaces $\mathcal{F}$ be a local system of spectra on X. We have a commutative diagram of fiber sequences

$$\begin{array}{ccccc} C^*(X, \partial X; \mathcal{F}) & \longrightarrow & C^*(X; \mathcal{F}) & \longrightarrow & C^*(\partial X; \mathcal{F} \mid \partial X) \\ \downarrow & & \downarrow & & \downarrow \\ C_*(X; \zeta_{(X,\partial X)} \wedge \mathcal{F}) & \longrightarrow & C_*(X, \partial X; \zeta_{(X,\partial X)} \wedge \mathcal{F}) & \longrightarrow & C_*(\partial X; (\Sigma \zeta_{(X,\partial X)} \wedge \mathcal{F}) \mid \partial X) \end{array}$$

where the vertical maps are given by cap product with $[X]$ and $[\partial X]$. The left vertical map is a homotopy equivalence by construction, and the right vertical map is a homotopy equivalence when $(X, \partial X)$ is a Poincare pair. It follows that if $(X, \partial X)$ is a Poincare pair, then the middle map is also a homotopy equivalence: that is, the cap product map

$$C^*(X; \mathcal{F}) \to C_*(X, \partial X; \zeta_{(X,\partial X)} \wedge \mathcal{F})$$

is a homotopy equivalence.

Suppose that $i : \partial X \to X$, and let R be an A_∞-ring with involution. We have a visible symmetric signature $\sigma^{vs}_{\partial X} \in \mathbb{L}^{vs}(\partial X, \zeta_{\partial X}, R)$, given by $(\underline{R}, q)$. Then q determines a symmetric bilinear form q_∂ on the object $i_! \underline{R} \in \mathrm{Shv}_{\mathrm{lc}}(X; \mathrm{RMod}_R)$ with respect to the quadratic functor $Q_{\Sigma \zeta_{(X,\partial X)}}$. We have a canonical map $i_! \underline{R} \to \underline{R}$, and a fiber sequence

$$C_*(X, \partial X; \zeta_{(X,\partial X)} \wedge \underline{R}^{h\Sigma_2}) \to C_*(X; \Sigma \zeta_{(X,\partial X)} \wedge i_! \underline{R}^{\mathrm{h}\Sigma_2}) \to C_*(X; \Sigma \zeta_{(X,\partial X)} \wedge \underline{R}^{\mathrm{h}\Sigma_2}).$$

Consequently, the fundamental class $[X]$ provides a nullhomotopy of the image of q_∂ in $Q_{\Sigma \zeta_{(X,\partial X)}}(\underline{R})$. This nullhomotopy exhibits $\underline{R}$ as a Lagrangian for the Poincare object $(i_! \underline{R}; q_\partial)$. In other words, it gives a canonical lifting of $\sigma^{vs}_{\partial X}$ to the homotopy fiber of the map

$$\mathbb{L}^{vs}(\partial X, \zeta_{\partial X}, R) \to \mathbb{L}^{vs}(X, \Sigma \zeta_{(X,\partial X)}, R).$$

Let us denote this lifting by σ^{vs}_X. We will refer to it as the *visible symmetric signature of* X (or the *visible symmetric signature* of the Poincare pair $(X, \partial X)$).

Notation 8. Let $f : Y \to X$ be a map of spaces and let ζ be a spherical fibration on X. We let $\mathbb{L}^{vs}(X, Y, \zeta, R)$ denote the homotopy cofiber of the map

$$\mathbb{L}^{vs}(Y, f^*\zeta, R) \to \mathbb{L}^{vs}(X, \zeta, R).$$

Equivalently $\mathbb{L}^{vs}(X, Y, \zeta, R)$ is the homotopy fiber of the map

$$\mathbb{L}^{vs}(Y, f^*\Sigma\zeta, R) \to \mathbb{L}^{vs}(X, \Sigma\zeta, R).$$

The upshot of the above discussion is that if $(X, \partial X)$ is a Poincare pair, we can identify σ_X^{vs} with a point in the 0th space of $\mathbb{L}^{vs}(X, \partial X, \zeta_{(X,\partial X)}, R)$. When $\partial X = \emptyset$, this specializes to the definition of the visible symmetric signature of a Poincare space described earlier.