Introduction (Lecture 1)

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A major goal of algebraic topology is to study topological spaces by means of algebraic invariants (such as homology or cohomology). There is a balance to be struck here: we would like our invariants to be simple enough to be tractable and computable, but rich enough to convey interesting information about topology. In this course, we are going to study one example where both of these demands can be satisfied. This is the so-called "chromatic" picture of stable homotopy theory, and it begins with Quillen's work on the relationship between cohomology theories and formal groups.

Let E be a multiplicative cohomology theory. For any topological space X, one can attempt to compute the E-cohomology groups $E^*(X)$ by means of the Atiyah-Hirzebruch spectral sequence

$$\mathrm{H}^p(X; E^q(*)) \Rightarrow E^{p+q}(X).$$

If X is the infinite dimensional projective space $\mathbf{C}\mathrm{P}^\infty$, then its ordinary cohomology groups are given by $\mathrm{H}^*(X; \mathbf{Z}) \simeq \mathbf{Z}[t]$, where $t \in \mathrm{H}^2(X; \mathbf{Z})$ is a generator. We say that E is *complex-orientable* if the Atiyah-Hirzebruch spectral sequence degenerates at the second page. In this case, we get an isomorphism $E^*(\mathbf{C}\mathrm{P}^\infty) \simeq E^*(*)[[t]]$ for some generator $t \in E^2(*)$.

In ordinary cohomology, we can define $t \in \mathrm{H}^2(\mathbf{C}\mathrm{P}^\infty; \mathbf{Z})$ to be the first Chern class $c_1(\mathcal{O}(1))$, where $\mathcal{O}(1)$ denotes the universal line bundle on $\mathbf{C}\mathrm{P}^\infty$. Conversely, if we are given t then we can define the first Chern class in general, using the fact that $\mathbf{C}\mathrm{P}^\infty$ is a classifying space for complex line bundles. Namely, if $\mathcal{L}$ is *any* complex line bundle on a (nice) space X, then there exists a continuous map $f : X \to \mathbf{C}\mathrm{P}^\infty$ (well-defined up to homotopy) and an isomorphism $\mathcal{L} \simeq f^* \mathcal{O}(1)$. We can then define $c_1(\mathcal{L}) = f^* t \in \mathrm{H}^2(X; \mathbf{Z})$.

If E is a complex-orientable cohomology theory, then the isomorphism $E^*(\mathbf{C}\mathrm{P}^\infty) \simeq E^*(*)[[t]]$ permits us to define a Chern class which takes values in E-cohomology. Namely, if $\mathcal{L}$ is a line bundle on a space X and $f : X \to \mathbf{C}\mathrm{P}^\infty$ is defined as above, then we can define $c_1^E(\mathcal{L}) = f^* t \in E^2(X)$.

Warning 1. The definition of the Chern class $c_1^E(\mathcal{L})$ depends not only on the cohomology theory E, but also on the choice of isomorphism $E^*(\mathbf{C}\mathrm{P}^\infty) \simeq E^*(*)[[t]]$ (that is, on the choice of t). A complex-orientable cohomology theory E together with a choice of generator $t \in E^2(\mathbf{C}\mathrm{P}^\infty)$ is called a *complex-oriented cohomology theory.*

We now ask: how well-behaved is this theory of E-valued Chern classes? For example, ordinary Chern classes satisfy a multiplicativity formula

$$c_1(\mathcal{L} \otimes \mathcal{L}') \simeq c_1(\mathcal{L}) + c_1(\mathcal{L}').$$

Does the analogous formula hold in E-cohomology? Generally, the answer is no. However, we can say that there is always *some* formula which allows us to express $c_1^E(\mathcal{L} \otimes \mathcal{L}')$ in terms of $c_1^E(\mathcal{L})$ and $c_1^E(\mathcal{L}')$. To see this, it suffices to consider the universal example of a space with two complex line bundles. This is the space $\mathbf{C}\mathrm{P}^\infty \times \mathbf{C}\mathrm{P}^\infty$. Using the Atiyah-Hirzebruch spectral sequence, we get an isomorphism $E^*(\mathbf{C}\mathrm{P}^\infty \times \mathbf{C}\mathrm{P}^\infty) \simeq E^*(*)[[u, v]]$. Here u and v denote the pullbacks of $t \in E^2(\mathbf{C}\mathrm{P}^\infty)$ along the two projection maps $\pi_1, \pi_2 : \mathbf{C}\mathrm{P}^\infty \times \mathbf{C}\mathrm{P}^\infty \to \mathbf{C}\mathrm{P}^\infty$; in other words, we can identify u and v with the Chern classes of the universal line bundles $\pi_1^* \mathcal{O}(1)$ and $\pi_2^* \mathcal{O}(1)$ on $\mathbf{C}\mathrm{P}^\infty \times \mathbf{C}\mathrm{P}^\infty$. We also have a third line bundle $\mathcal{O} = \pi_1^* \mathcal{O}(1) \otimes \pi_2^* \mathcal{O}(1)$. Then

$c_1^E(\mathcal{O}) = f(u,v) \in E^*(\mathbf{C}\mathrm{P}^\infty \times \mathbf{C}\mathrm{P}^\infty) \simeq E^*[[u,v]]$. By the method of the universal example, we deduce that $c_1^E(\mathcal{L} \otimes \mathcal{L}') \simeq f(c_1^E \mathcal{L}, c_1^E \mathcal{L}')$ for any pair of line bundles $\mathcal{L}$ and $\mathcal{L}'$ on any space X (here we must take some care with the meaning of this statement, since f is a power series and not a polynomial in general).

Example 2. For the usual theory of Chern classes, f is given by the formula $f(u,v) = u + v$.

What can we say about the power series f? In general it is a power series rather than a polynomial, and can be quite complicated. However, it is not arbitrary: it satisfies certain identities, which reflect the idea that the tensor product of complex line bundles is commutative and associative up to isomorphism. More precisely, we have

$$f(u,0) = u = f(0,u)$$

$$f(u,v) = f(v,u)$$

$$f(u, f(v,w)) = f(f(u,v), w).$$

In general, if R is a commutative ring, then a power series $f(u,v) \in R[[u,v]]$ satisfying the identities above is called a *formal group law* over R.

We can summarize our discussion as follows: every complex-oriented cohomology theory E determines a formal group law over the commutative ring $E^{even}(*)$. This assignment fits into the general paradigm of algebraic topology. A cohomology theory E should be regarded as a topological object: it can be represented by a spectrum, which is a variation on the notion of a space. To this cohomology theory we assign an algebraic object: a formal group law over a commutative ring. This assignment satisfies both of the requirements posited at the beginning of this lecture:

(a) Though somewhat more complicated than an abelian group or a vector space, a formal group law is a reasonably tractable mathematical object. In particular, formal group laws have been thoroughly studied by algebraic geometers and number theorists.

(b) The formal group law associated to a complex-oriented cohomology theory E remembers a great deal about E. In fact, one can often reconstruct E from its formal group law.

To elaborate on these points, we first note that there is a *universal* example of a formal group law. That is, there is a commutative ring L and a formal group law $f(u,v) \in L[[u,v]]$ which is "maximally complicated", in the sense that any other formal group law over a commutative ring R is obtained from $f(u,v)$ by means of a ring homomorphism $L \to R$. The ring L is called the *Lazard ring* in honor of Lazard, who proved that L is a polynomial ring (in infinitely many generators).

According to a theorem of Quillen, the Lazard ring L has another incarnation: it is the coefficient ring of the cohomology theory MU of *complex bordism* (which is *universal* among complex-oriented cohomology theories). One can attempt to use this observation to construct an "inverse" to the above constructions. Namely, suppose we are given a commutative ring R and a formal group law $f(u,v) \in R[[u,v]]$, classified by a map $L \to R$. We can then attempt to define a new cohomology theory E (having coefficient ring R) by the formula $E^*(X) \simeq MU^*(X) \otimes_L R$ for finite complexes X (for this to be sensible, R should be equipped with a suitable grading; we will suppress mention in the discussion which follows). This construction does not always work: that is, E^* does not always have the excision and Mayer-Vietoris exact sequences that are required of cohomology theories. However, a fundamental result of Landweber gives a purely algebraic criterion on ϕ which, if satisfied, guarantees that E^* is a cohomology theory. One can use this criterion to produce many interesting examples of cohomology theories.

Example 3. One can take R to be the ring of integers and $f(u,v)$ to be the *multiplicative* formal group given by $f(u,v) = u + v + uv$. In this case, Landweber's theorem applies and produces a cohomology theory, namely, complex K-theory.

Motivated by Example 3, it is natural to ask what other cohomology theories can be produced by means of Landweber's theorem: that is, starting with a map of affine schemes $\phi : \operatorname{Spec} R \to \operatorname{Spec} L = \mathbf{A}^\infty$. First, we should note that the map ϕ is not really fundamental. The formal group law associated to a cohomology theory E depends not only on E, but also on a choice of complex orientation $t \in E^2(\mathbf{CP}^\infty)$. The collection of all such choices is acted on by the group G of coordinate changes

$$t \mapsto t + a_1 t^2 + a_2 t^3 + ...$$

Consequently, our real interest is not in the moduli space $\operatorname{Spec} L$ of formal group laws, but in the quotient $\operatorname{Spec} L/G$. This is a kind of algebraic stack, called the *moduli stack of formal groups* (more precisely, it is the moduli stack of formal groups with trivialized Lie algebras). The main thrust of this course can be stated as follows:

- The structure of the stable homotopy category is controlled by the geometry of the stack $\operatorname{Spec} L/G$.

For example, every complex-orientable cohomology theory E determines a commutative ring $R = E^{even}(*)$ and a formal group over R, which we can think of as a map $\operatorname{Spec} R \to \operatorname{Spec} L/G$. This construction provides the beginning of a rough dictionary:

Multiplicative cohomology theories	Affine schemes over $\operatorname{Spec} L/G$
Cohomology theories	Quasi-coherent sheaves on $\operatorname{Spec} L/G$
Complex bordism	Moduli Space $\operatorname{Spec} L \simeq \mathbf{A}^\infty$ of formal group laws
...	...

As we will see over the course of the semester, these ideas give an extremely useful picture of the stable homotopy category.