

ELEMENTS OF FINITE GEOMETRY

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Unit 4: Euler's Gem

A q -sphere G is a q -manifold which when punctured becomes contractible. Euler's Gem formula tells that Euler characteristic is $\chi(G) = 1 + (-1)^q$. We can classify all Platonic q -spheres, inductively defined as q -spheres for which all unit spheres are isomorphic and equal to some Platonic $(q - 1)$ -sphere.

4.1. A graph (V, E) is **contractible** if there exists $v \in V$ such that $S(v)$ and $G \setminus v$ are both contractible. The 1-point graph $1 = K_1$ is contractible. The empty graph is not contractible.

4.2. The empty-graph is the (-1) -sphere. A graph G is called a **q -manifold** if all unit spheres $S(v)$ are $(q - 1)$ -spheres. A **q -sphere** is a q -manifold for which there is a v such that $G \setminus v$ is contractible.

4.3. Every contractible graph has $\chi(G) = 1$ by induction: $\chi(G) = \chi(G \setminus v) + \chi(S(v)) - \chi(\{v\}) = 1 + 1 - 1 = 1$. If A, B are graphs $A \cap B$ carries the intersection of the simplicial complexes of A and B , then the **valuation formula** for simplicial complexes holds also for graphs $\chi(A) + \chi(B) - \chi(A \cap B) = \chi(A \cup B)$. $A \cup B$ carries the intersection of the simplicial complexes of A and B . The Euler Gem formula is:

Theorem: $\chi(G) = 1 + (-1)^q$ for a q -sphere.

Proof. (i) In every graph, every unit ball $B(v)$ is contractible. It follows that $\chi(B(v)) = 1$. (ii) for a q -sphere, every unit sphere $S(v)$ is a $(q - 1)$ -sphere and so has by induction the Euler characteristic $1 + (-1)^{q-1} = 1 - (-1)^q$. (iii) Now use $\chi(G) = \chi(G \setminus v) + \chi(B(v)) - \chi(S(v)) = 1 + 1 - (1 + (-1)^{q-1}) = 1 + (-1)^q$. We used the valuation formula. $\square$

4.4. The **join** of two graphs $A \oplus B$ has vertex set $V(A) \cup V(B)$ and edge set $E(A) \cup E(B) \cup \{(a, b), a \in V(A), b \in V(B)\}$. It is the dual of disjoint union $+$ because $A \oplus B = \overline{A + B}$. With this operation and the zero element 0 we have a second monoid, dual and isomorphic to the monoid with disjoint union. Spheres form a sub-monoid:

Theorem: A is a k -sphere, B is a l -sphere $\Rightarrow A \oplus B$ is a $(k + l + 1)$ -sphere.

Proof. Use induction with respect to $k + l = n$. For $n = 0$ meaning $k = l = 0$, the join is a 1-sphere. Assume we have proven it for all cases with $k + l \leq n - 1$ and assume $k + l = n$. By definition every unit sphere $S_A(v)$ is a $(k - 1)$ -sphere and every unit sphere $S_B(w)$ is a $(l - 1)$ -sphere. For $v \in V(A)$, the graph $S_{A \oplus B}(v) = S_A(v) \oplus B$ is a $l - 1 + k + 1$ -sphere by induction. For $w \in V(B)$, the graph $S_{A \oplus B}(w) = A \oplus S_B(w)$ is a $l + k - 1 + 1$ -sphere by induction. This shows that $A \oplus B$ is a $l + k + 1$ manifold. Now use that if A is contractible and B is arbitrary than $A + B$ is contractible. This shows that for $v \in V(A)$, the graph $A \oplus B \setminus v = (A \setminus v) \oplus B$ is contractible. Similarly, for $w \in V(B)$, the graph $A \oplus B \setminus w = A \oplus (B \setminus w)$ is contractible. $\square$

4.5. Inductively, a q -sphere is declared to be **Platonic**, if all unit spheres are Platonic $(q - 1)$ -spheres that are all isomorphic. There is a unique Platonic sphere in all dimensions except dimensions 1,2,3. There are infinitely many 1-spheres C_n and they are all Platonic. For $d = 2$, there is the **octahedron** and **icosahedron**. For $d = 3$, there is the **600 cell** and the **16 cell**. After that there are only cross polytopes, the 2^{q+1} cell.

Theorem: There is a unique Platonic sphere for $q > 3$.

Proof. $q = -1, 0, 1$ are clear. For $q = 2$, the curvature $K(x) = 1 - f_0/2 + f_1/3 - f_2/3$ is constant, adding up to 2. It is either $1/3$ or $1/6$. For $d = 3$, where each $S(x)$ must be either the octahedron or icosahedron, G is the 16 cell or 600 cell. For $d = 4$, by Gauss-Bonnet, $K(x)$ add up to 2 and be of the form $L/12$ for some integer. For $L = 1$, there exists the 4-dimensional cross polytope with f -vector $(10, 40, 80, 80, 32)$. There is no 4-sphere, for which $S(x)$ is the 600-cell. As the f -vector of it is $(120, 720, 1200, 600)$, we would get $K(x) = 1 - 120/2 + 720/3 - 1200/4 + 600/5 = 1$ requiring $|V| = 2$ and $\dim(G) \leq 1$. Inductively there is now only one Platonic q -sphere for $q > 3$. $\square$

4.6. When classifying **Platonic solids**, one traditionally adds the dual polytopes which are not q -manifolds as they are triangle free. Classics always also would add the $(q+1)$ -simplices because their boundary simplicial complex are $(q - 1)$ -spheres as simplicial complexes. The 24-cell U , the units of the Hurwitz quaternions is the only additional polytop. One could prove this by extending Gauss-Bonnet to combinatorial cell-complexes: inductively add k -cells to an already existing $(k - 1)$ sphere. The notion of q -sphere, q -manifold, contractibility and Platonic sphere are in the same way generalized to such discrete complexes. This purely combinatorial approach is analog to CW complexes in the continuum. After extending Gauss-Bonnet in the same way, build up the Platonic spheres inductively by dimension. U does not occur as a unit sphere of any Platonic 4-complex. That completes the classical story. Traditionally, the continuum is involved: regular q -polytopes are boundaries of convex regions in $\mathbb{R}^{q+1}$.

4.7. Examples: 1) Cross polytopes $K_{2,2,\dots,2}$ are the n -fold suspension of 0. $K_{\{\}} = 0$, $K_{\{2\}} = \overline{K_2} = S^0$, $K_{\{2,2\}} = K_{2,2} = C_4$. $K_{2,2,2} = S_0 \oplus S_0 \oplus S_0 = O$ is the octahedron. $K_{2,2,2,2} = C_4 \oplus C_4$ is the smallest 3-sphere. 2) The join of two icosahedra is a 5-sphere. But it is not Platonic.