A SUMMER OF GEOMETRY

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Appendix 4: Topology

4.1. A **topological space** $(X, \mathcal{O})$ is a set X with a collection $\mathcal{O}$ of subsets of X with the property that both $\emptyset$ and X are in $\mathcal{O}$ and that $\mathcal{O}$ is closed under finite intersections and closed under arbitrary unions. A topological space is **finite**, if X is a finite set. A **base** $\mathcal{B}$ of a topological space is a subset of $\mathcal{O}$ that generates $\mathcal{O}$ in the sense that every $A \in \mathcal{O}$ can be written as a union of base elements, where the empty union $\bigcup_{A \in \emptyset} A = \emptyset$. The base does not have to contain the empty set $\emptyset$.

4.2. The **Alexandroff topology** is the topology generated by the base $\{U(x)\}_{x \in G}$, given by the **stars** $U(x) = \{y \in G, x \subset y\}$. A **sub-base** is $\{U(v)\}_{x \in V = G_0}$ because intersections of the subbase produce base elements. If G has positive maximal dimension, the topology is non-Hausdorff. Given an edge (a, b), the points a, b can not be separated by open sets as every open $U(a) \cap U(b) = U((a, b))$ is open and not-empty.

4.3. An arbitrary topology is called **Alexandroff**, if arbitrary intersections of open sets are open. Every finite topology is automatically Alexandroff. If the abstract simplicial complex is not finite but has a finite maximal dimension, then still, its topology is Alexandroff.

4.4. A topology $(G, \mathcal{O})$ is **Kolmogorov** (T0) if for any two distinct points $x, y \in G$, there exist neighborhoods $U(x), U(y)$ such that either $x \notin U(y)$ or $y \notin U(x)$. A topology is **Fréchet** (T1) if for any two distinct points $x, y \in G$, there exist neighborhoods $U(x), U(y)$ such that $x \notin U(y)$ and $y \notin U(x)$. A topology is **Hausdorff** (T2) if for any two distinct points $x, y \in G$, there exist neighborhoods $U(x), U(y)$ such that $U(x) \cap U(y) = \emptyset$.

4.5. The topology on a simplicial complex G is always **Kolmogorov**. Proof: given $x \neq y \in G$. If there is an inclusion like $x \subset y$, then $U(y) \subset U(x)$ but $x \notin U(y)$. If there is no inclusion, then both $x \notin U(y)$ and $y \notin U(x)$. But a topology on G is neither **Fréchet**, nor **Hausdorff** if its maximal dimension is positive. If $x \subset y$, then y is always in $U(x)$. As any finite topology, it is **Alexandroff**, meaning that there are smallest neighborhoods $U(x) = \{y \in g, x \subset y\}$ of every $x \in G$. The topology is **Zariski type** because the closed sets are the sub-simplicial complexes. (Associate closed sets with varieties. A finite topological space that is Hausdorff is necessarily the discrete topology, in which all subsets are open. Finite topologies are never Hausdorff unless they are zero-dimensional.

4.6. Define the **unit ball** $B(x) = \overline{U(x)}$ and the **unit sphere** $S(x) = B(x) \setminus U(x)$. The later is the boundary of the unit ball and is again closed as the intersection of a closed set $B(x)$ with a closed set $U(x)^c$. If $x = v$ is 0-dimensional, the unit sphere $S(x)$ is sometimes called the **link**.

4.7. Note that G is a set of sets and that elements in $\mathcal{O}$ are a set of subsets of G. The open sets in $G = \{\{1\},\{2\},\{1,2\}\}$ are $\{\emptyset, \{\{1\},\{1,2\}\}, \{\{2\},\{1,2\}\}, \{\{1,2\}\}, G\}$. So, it is important to distinguish a simplex $x \in G$ with $\{x\}$ which is in general neither open nor closed. Maximal simplices x produce the open set $U(x) = \{x\}$, vertices x produce closed sets $C(x) = \{x\}$. If x is an intermediate dimension, then $\{x\}$ is neither open nor closed. The smallest closed set containing x is $\overline{\{x\}}$, the smallest open set containing x is $U(x)$.

4.8. One could also look at topologies on V but that is a different thing. For example, if G is the complex of K_4 with vertex set $V = \{1,2,3,4\}$, then the pseudo circle $\mathcal{O} = \{(1,2,3,4),(1,2,3),(1,2,4),(1,2),(1),(2),(0)\}$ a subset of G, not a set of sets from G. It is known as the small circle. It is neither an open set nor a closed set in G.

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Generate[A_]:=Sort[Delete[Union[Sort[Flatten[Map[Subsets,Map[Sort,A]],1]]],1]];
Whitney[s_] :=Map[Sort,Generate[FindClique[s,Infinity,All]]];
ToGraph[G_]:=Graph[n=Length[G];
  Select[Flatten[Table[G[[k]]<->G[[l]],{k,n},{l,k+1,n}],1],(SubsetQ[#[[2]],#[[1]]])&]];
G=Sort[{{1,2,3,4},{1,2,3},{1,2,4},{1,2},{1},{2}}];
G=Generate[{{a,b,c},{a,b,d}}]
GraphPlot[ToGraph[G],GraphStyle -> "SmallNetwork"]
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