A SUMMER OF GEOMETRY

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Appendix 2: Simplicial complexes

2.1. A **finite abstract simplicial complex** G is a finite set of non-empty sets that is closed under the operation of taking non-empty subsets. This geometric structure has only one axiom, the **hereditary axiom**. This simplicity makes it appealing. There is a prise to pay however. Sets of sets are less intuitive than graphs.

2.2. As pointed out in the appendix about graphs, we can always associate a graph Γ to a simplicial complex. This graph has the sets as vertices and connects two if one is contained in the other. There is also a "reverse" operation: given a graph, we can look at its simplicial complex, the Whitney complex in which the vertex sets of complete subgraphs are the sets. The maximal dimension of the simplex is the maximal dimension of the graph.

2.3. Given a simplicial complex, can assume that every $x \in G$ comes with an order. The choice of the order corresponds to picking a coordinate system and is irrelevant. It does not matter what order we chose for each x, but it can be important to have an order in order to do linear algebra. We do not insist that the order is inherited by subsets. There is no order on maximal simplices in general which works like that, the simplest example being the Moebius strip complex, which is the simplicial complex of the graph complement of C_7. If G contains subsets of integers, we can take for example the natural order. We also assume that the elements of G are ordered according to dimension to get block matrices. We for example write $G = \{\{1\},\{2\},\{1,2\}\}$ the complex of K_2 and not $G = \{\{1,2\},\{1\},\{2\}$. Again, this is just a choice of labeling a basis when doing linear algebra and irrelevant for any geometry. Not that one could look at the order it is just that we are not interested in any mathematics where the order matters.

2.4. As custom in set theory, sets contain different elements. $G = \{\{1,2\},\{1\},\{2\}\}$ is an example of a simplicial complex. The structure $G = \{\{1\}.,\{1,1\}\}$ would be identified with $\{\{1\},\{1\}\}$ and so with $\{\{1\}\}$. A set does not record multiplicities. If this is needed, like for multi-graphs, we would use sequences. The sequence $(\{1\},\{2\},\{1,2\},\{1,2\})$ for example is a quiver with two edges connecting 1 with 2.

2.5. An equivalent definition of simplicial complexes is more clumsy but it is often seen in the literature: one defines a simplicial complex by invoking a **vertex set** $V = \bigcup_{x \in G} x$. The assumption is then that G is a subset of $2^V \setminus \{\{\}\}$ that does not contain the empty set and that satisfies the heriditary axiom. In that frame work, a simplicial complex is a pair (V,G) where $G \subset 2^V$. It gives away however one of the conceptional advantages of simplicial sets, its "simplicity", having only one axiom. V can be derived from G, the earlier definition is slightly simpler and equivalent.

2.6. More general than a simplicial complex is a set of sets. This structure has also been called a **hypergraph**. A hypergraph in general does not allow a reasonable calculus. A hypergraph also is allowed to contain the empty set. In order for the theory to work properly, it is important not to let the empty set to be in a simplicial complex. The heriditary assumption is needed to get a notion of derivative that works.

2.7. There are flavors of combinatorial topology that include $\emptyset$ into G. The **Euler characteristic** $\chi(G) = \sum_{x \in G} \omega(x)$ with $\omega(x) = (-1)^{\dim(x)}$ for example would not work. [1] There are compelling reasons to keep $\chi(G)$ the Euler characteristic like compatibility with addition (disjoint union) and multiplication (Cartesian product). There are compelling reasons to keep simplices to be contractible objects for which the closure $\overline{\{x\}}$ is a simplicial complex of Euler characteristic 1 and that spheres are non-contractible objects of Euler characteristic $1+(-1)^q$. The void is the (-1) sphere. It has Euler characteristic 0 and is not contractible.

2.8. Also in **matroid theory**, there are flavors of the theory that include the empty set and versions, where the empty set is excluded. If the empty set $\emptyset$ is excluded in a matroid G, one can say a matroid is a simplicial complex which additionally satisfies the **exchange axiom**: if $|y| < |x|$ then some $v \in x \setminus y$ satisfies $y \cup \{v\} \in G$. An example of a simplicial complex that is not a matroid would be the path complex $G = \{\{1,2,3\},\{1,2\},\{2,3\},\{1\},\{2\},\{3\}\}$. Since matroid theory had been developed however in the context of linear algebra first, one usually assumes that a matroid contains the empty set.

2.9. To drill this a bit further: in linear algebra, one calls a subset V of a vector space a **linear space** if it is compatible with addition, scalar multiplication and if it contains the 0 element. One could argue to simplify that axiom system by skipping the requirement to have 0 in the reason being that if $x \in V$ and $\lambda = 0$ implies by the compatibility with scalar multiplication that $\lambda x = 0 \in V$. The requirement that 0 must be in V seems redundant. But it is not: if we take that axiom away, the empty set $V = \emptyset$ would be considered a linear space, but most linear algebra texts assumes this not to be the case. Also in algebra, the 1-point structure $\{0 = 1\}$ (the trivial ring) with trivial addition and multiplication $0+0=0, 0*0=0$ is not always considered a "ring with 1". The Boolean ring on the empty set is the trivial ring.

2.10. Simplicial complexes can be defined from other structures. Any graph defines a **Whitney complex**, in which the vertex sets of complete subgraphs are the sets. It is also called the **flag complex** or **order complex**. The empty graph generates the void. It is not a complete complex as a simplicial complex never contains the empty set. An other complex is the **forest complex**, in which the edges of a forest are the sets. A forest decomposes into trees and trees are assumed in this set-up to be one-dimensional, meaning to have maximal dimension 1. Single points = seeds, are not yet considered trees. In other contexts, like for the matrix forest theorem, it is helpful to include seeds as trees.

[1] Extended Euler characteristic $\chi(G) - 1$ is popular as it renders the join multiplicativ.

2.11. Simplicial complexes can be derived from already given simplicial complexes. The intersection of two simplicial complexes is a simplicial complex. If two simplicial complexes are disjoint, then their intersection is the void, which is a simplicial complex. The union of two simplicial complexes $G = A \cup B$ is a simplicial complex. Also this is clear from the definition. If $x \in G$, then either $x \in A$ or $x \in B$ so that the hereditary axiom holds. If G is a simplicial complex of dimension q, the **skeleton complex** of dimension $k \leq q$ is the set of sets for which the dimension is smaller or equal than k.

2.12. Given an arbitrary set A of non-empty sets, one can look at the closure $\overline{A}$ of A. It is the smallest simplicial complex containing A. To get $\overline{A}$, just adjoin all the non-empty subsets of A. The name closure is adequate because it is the closure operation in the Alexandroff topology, the topology first considered by Pavel Alexandrov. [2]

2.13. The set of finite abstract simplicial complexes forms a category in which the objects are the simplicial complexes and the morphisms are simplicial maps $f : A \to B$. A **simplicial map** maps $V(A)$ to $V(B)$ and preserves the order. If $x \subset y$, then $f(x) \subset f(y)$. If f is invertible and the inverse is a simplicial map, then we have a simplicial isomorphism. Maximal dimension can only decrease under simplicial maps. A version of Cantor-Schröder-Bernstein assures that if there are injective simplicial maps from A to B and from B to A, then A, B are isomorphic. The injectivity assumption assures that $V(A), V(B)$ have the same cardinality. The bijection then defines a bijection from A to B as simplices are mapped to simplices.

```
Generate[A_]:=Sort[Delete[Union[Sort[Flatten[Map[Subsets,Map[Sort,A]],1]]],1]];
Whitney[s_] :=Map[Sort,Generate[FindClique[s,Infinity,All]]];
EulerChi[G_]:=Sum[-(-1)^Length[G[[k]]],{k,Length[G]}];
Fvector[s_]:=Delete[BinCounts[Map[Length,Whitney[s]]],1];

(* recursive inductive computation of the simplicial complex of a graph *)
UnitSphere[s_, v_]:=VertexDelete[NeighborhoodGraph[s, v], v];
q[X_, v_]:=Table[Sort[Append[X[[k]], v]],{k,Length[X]}];
W[s_]:=Module[{V,A,n},V=VertexList[s];n=Length[V]; A=Table[{V[[k]]},{k,n}];
 If[n==0,{},Union[A,Flatten[Table[q[W[UnitSphere[s,V[[k]]]],V[[k]]],{k,n}],1]]]];
```

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[2] Alexandrov is the common spelling for the person, Alexandroff is the common use for mathematical results.