A SUMMER OF GEOMETRY

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Appendix 1: Sets

1.1. Finite sets are an elementary topos that extends to a commutative ring for which the cardinality function produces a ring homomorphism to the integers. Every finite set G of sets also defines a Boolean ring in which the symmetric difference is the addition and the intersection is the multiplication. The zero element is the empty set, the 1-element is the union $V = \bigcup_{x \in G} x$.

1.2. A **finite set** is a collection G of finite objects. It is a standard assumption in set theory that a set does not record multiplicities. The object $\{a, a, b, a, b\}$ for example is identified with $\{a, b\}$ if it is considered to be a set. If multiplicity should be needed, one could write it as a multi-set or a function $\{a, b\} \to \mathbb{N}$ encoding the multiplicities. If the order should matter, one could write it as a finite sequence $\{a, a, b, a, b\}$ or word $aabab$ over an alphabet.

1.3. If V is a finite set, $G = 2^V$ is the set of all subsets of V. This set has a **partial order structure** $A \leq B$ in the form of $A \subset B$. It is also standard in set theory not to assume the subset operation to be strict. The statement $A \subset A$ for example is always true. The set 2^V contains the empty set $\emptyset$. If $V = \{\} = \emptyset$ is the empty set, then $2^V = \{\emptyset\} = 1$ is a set with one element. Let $|A|$ denote the **cardinality** of the set. The cardinality is a non-negative integer $\mathbb{N} = \{0, 1, 2, 3, \dots\}$. We would write $\mathbb{N}^+$ for the positive integers and then there are the integers $\mathbb{Z}$, the group completion of $\mathbb{N}$. We have $|0| = 0, |1| = 1$ and $|2^A| = 2^{|A|}$ justifying the notation. There is a natural identification of 2^{A+B} with the Cartesian product $2^A \times 2^B$ because if x is a subset of $A + B$, then x defines (a, b) with $a = x \cap A$ and $b = x \cap B$. But $(a, b) \in 2^A \times 2^B$.

1.4. Finite sets form a **category** FinSet in which the finite sets are the **objects** and where maps $A \to B$ are the arrows. The morphisms are just functions and the composition of arrows is the composition of functions. The isomorphisms in FinSet are the bijective maps. The empty set 0 is the **initial object**, the singleton set 1 is the **terminal object**. The category FinSet is an **elementary topos**. It has **finite limits**, **exponentials** A^B satisfying $A^{B+C} \sim A^B \times A^C$ and has a **subobject classifier** given by the **characteristic function** $x \to 1_A(x) \in \{0, 1\}$ encoding whether elements are in A or not.

1.5. There is a monoid operation on the objects of FinSet. It is the disjoint union $+$, that is also called the **coproduct**. It is associative and so a **monoid structure**. The empty set $\emptyset$ is the zero element. If this monoid is group completed to become a group, one has then not only sets but "negative sets". If the coproduct is **group completed** with negative sets, one has an additive group.

1.6. There is also a multiplication $A, B \to A \times B$ called the **Cartesian product**. $A \times B$ is defined as the set of all ordered pairs $\{(a,b), a \in A, b \in B\}$. The singleton set 1 is the 1-element in this multiplication. The **Cartesian ring** $(FinSet, +, 0, \times, 1)$ is a commutative ring with 1 if one identifies $(-A) \times B = A \times (-B) = -(A \times B)$. The category FinSet so naturally extends to a commutative ring in which 1 is the one-element and where 0 is the zero-element. If the cardinality function is extended to negative sets as $|-A| = -|A|$, the cardinality function defines a homomorphism from the ring of finite sets to $\mathbb{Z}$. If sets of the same cardinality are identified, then this is an isomorphism.

1.7. In the later, the symmetric difference $A+A$ always has an even number of elements so that odd sets are additive primes and $A * A = A$. In the Cartesian ring, no set of prime cardinality can be written as $A \times B$ unless $A = 1$ or $B = 1$. Sets of prime cardinality are multiplicative primes in the Cartesian ring.

1.8. There is a completely different ring structure if a non-empty host set V is given. A set of sets G that is closed under intersection $A \cap B$ and symmetric difference $A+B = (A \cup B) \setminus (A \cap B)$ and contains $1 = V$ is a **Boolean algebra**. We always have $A*A = A, A+A = 0$. This commutative ring is the **Boolean ring** $(2^V, +, \cdot, 0, V)$. The empty set $0 = \emptyset$ is the zero-element and V is the one-element. The related structure $(V, \cup, \cap, 0, V)$, the **Boolean lattice**.

1.9. A set of non-empty sets is called a **simplicial complex** if it is closed under the operation of taking non-empty subsets. An arbitrary set of sets G generates a simplicial complex by including all non-empty subsets of sets in G. The empty set $\emptyset$ is a simplicial complex called the **void** but it is never assumed to be an element in any simplicial complex.

1.10. Set theory can be considered to be geometry on zero-dimensional structures. It is the mathematics of 0-dimensional simplicial complexes. The Cartesian product of simplicial complexes is not a simplicial complex. This will require to look either at delta sets of Barycentric refined products.

```
Union[{1,2,3},{2,3,4}]
SubsetQ[{1,2,3},{1,2,3}]
EmptyGraphQ[Graph[{},{}]]
Zero=Range[0]; One=Range[1];
SetProduct[A_,B_]:=Table[{A[[k]],B[[l]]},{k,Length[A]},{l,Length[B]}];
SetProduct[Range[3],Range[4]]
SetAdd[A_, B_]:=VertexList[GraphDisjointUnion[Graph[A,{}],Graph[B,{}]]];
SetAdd[{1,2,3}, {1,2,3,4}]
SetPower[A_,k_]:=Tuples[A,k]
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