A SUMMER OF GEOMETRY

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Appendix 0: Graphs

0.1. Graphs form a structure that is as powerful as simplicial complexes or delta sets. Any simplicial complex and every delta set defines a graph. The power of graphs is that they are accessible, visual and intuitive. They complement simplicial complexes which are simple and delta sets, which are general.

0.2. A **finite simple graph** is a pair (V, E), where V is a finite set and $E \subset \{\{a, b\}, a, b \in V, a \neq b\}$ is a set of pairs of points. Elements in V are called **vertices**. Elements in E are called **edges**. A **finite simple digraph** is a pair (V, E), where V is a finite set and $E \subset \{(a, b), a, b \in V, a \neq b\}$ is a set of oriented pairs of points. If both $(a, b), (b, a)$ are in E, the edge (a, b) is considered undirected. A **finite quiver** is a pair (V, E), where V is a finite set and E is a sequence of points in $V \times V$; elements $(a, a) \in E$ are called **loops**. A finite quiver is also called a **directed multigraph with loops allowed**. If a quiver is assumed to be undirected, it is also called a **undirected multigraph** or **double quiver** with loops allowed. If no loops are allowed, one needs to specify as the literature on multi-graphs and quivers handles the presence of loops differently. The **structures** "simple graph","simple digraph" and "quiver" are however unambiguous in almost all of the literature.

0.3. Given a subset W of V, the **graph generated by** W is the graph $(W, \{(a, b) \in E, a \in W, b \in W\})$. For example, if W is the set of vertices that are connected directly to a vertex v, it induces the **unit sphere** $S(v)$. For example: the empty set generates the empty graph. The vertex set generates the entire graph. If $(a, b) \notin E$, a pair of points (a, b) in V generates a 0-sphere, a zero dimensional graph without edges. If $(a, b) \in E$, the pair of points generates K_2.

0.4. A graph (V, E) comes naturally with a simplicial complex G, where G are the vertex sets of complete subgraphs. The **dimension** of the graph is the maximal dimension of G. A triangle-free graph has dimension 1. A graph without edges has dimension 0. The empty graph has dimension -1. A tetrahedron K_4 has dimension 3. If we want to talk about the surface of the tetrahedron, we have to look at the skeleton complex consisting of all simplices of dimension 2 or lower. So, if $G = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{3, 1\}, \{1, 2, 3\}\}$ is the complex of K_3, then its 2-skeleton complex is $H = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{3, 1\}\}$. A complex G is called contractible if there exists x such that $S(x) = \overline{U}(x) \setminus U(x)$ and $G \setminus U(x)$ are both contractible with the induction assumption that $1 = \{1\}$ is contractible. The complex G from K_4 is contractible, the skeleton complex H is not.

0.5. A finite abstract simplicial complex G defines a finite simple graph $\Gamma = (V, E)$ in which $V = G$ are the vertices and a pair (a, b) is in the edge set E, if $a \subset b$ or $b \subset a$. The complex G can be defined to be contractible if its graph is contractible. For simplicial complexes, there are more contraction steps: G is contractible, if there is a simplex $x \in G$ for which both the unit sphere $S(x) = \overline{U(x)} \setminus U(x)$ as well as $G \setminus U(x)$ are contractible simplicial complexes. If G is the Whitney complex of a graph $\Gamma = (V, E)$ and $x = v$ is a 0-dimensional simplex, then $G \setminus U(x)$ is the Whitney complex of $\Gamma \setminus v$.

0.6. If (V, E) is a graph, we can look at the graph of its complex G. This is called the **Barycentric refinement** of G. The Barycentric refinement of a complete graph K_{q+1} is a q-manifold with boundary. The Barycentric refinement of a triangle K_3 for example is a wheel graph with 7 vertices. If G is a complex, the Barycentric refinement is done by first looking at its graph Γ and then taking the Whitney complex of that graph. In any case, the combination of producing a graph from a complex and producing a complex from a graph is a Barycentric refinement.

0.7. Given two finite simple graphs A, B, the **join** is the graph $A \oplus B = (V, E)$, where $V = V(A) \cup V(B)$ and $E = E(A) \cup E(B) \cup \{(a, b), a \in V(A), b \in V(b)\}$ are the edges. The simplicial complex G of $A \oplus B$ is the join of the simplicial complexes of A and B. Every simplex $x \oplus y$ of G is a join of two simplices x and y. The f-vector of G encodes the cardinalities $f_k = |G_k|$ of k dimensional simplices in G. It defines the simplex generating function $f(t) = 1 + f_0 t + \cdots + f_q t^{q+1}$. We have $f_{A \oplus B}(t) = f_A(t) f_B(t)$. On the simplicial complex level, the join of two simplicial complexes G, H is the union $G \cup H \cup \{x \cup y, x \in G, y \in H\}$. If G, H were the Whitney complexes of graphs, then their join is the Whitney complex of the graph join.

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n=10; m=50; v=Range[n]; R:=RandomChoice[v]; e=Table[R->R,{m}];
quiver=Graph[e];
simpledigraph=Graph[Select[Union[Map[Sort,e]],#1[[1]]!=#[[2]]&]];
simplegraph=UndirectedGraph[simpledigraph];
Q[s_]:=GraphPlot[s,Rule[GraphLayout,"SpringElectricalEmbedding"],
                   Rule[GraphStyle,"SmallNetworks"]];
GraphicsRow[{Q[quiver],Q[simpledigraph],Q[simplegraph]}]
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