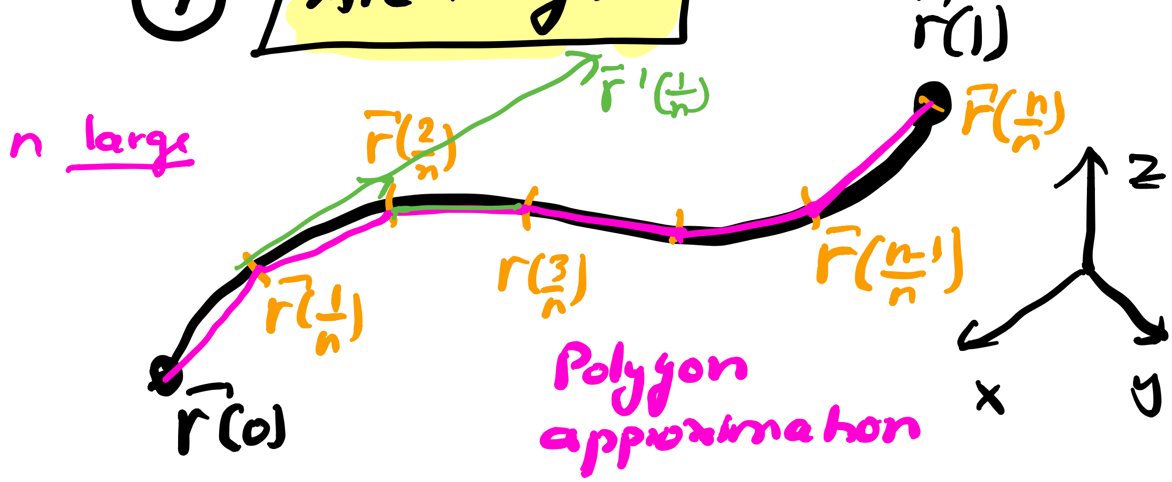
Unit 8

① Arc length



$$L = \left|\vec r(\tfrac{1}{n}) - \vec r(\tfrac{0}{n})\right| + \left|\vec r(\tfrac{2}{n}) - \vec r(\tfrac{1}{n})\right| + \dots$$

Length of polygon

$$\dots + \left|\vec r(\tfrac{n}{n}) - \vec r(\tfrac{n-1}{n})\right|$$

$$\sum_{k=0}^{n-1} \Big|\underbrace{\vec r(\tfrac{k+1}{n}) - \vec r(\tfrac{k}{n})}_{\sim \frac{1}{n}\left|\vec r'(\frac{k}{n})\right|}\Big|$$

$$= \frac{1}{n} \sum_{k=0}^{n-1} \left|\vec r'(\tfrac{k}{n})\right|$$

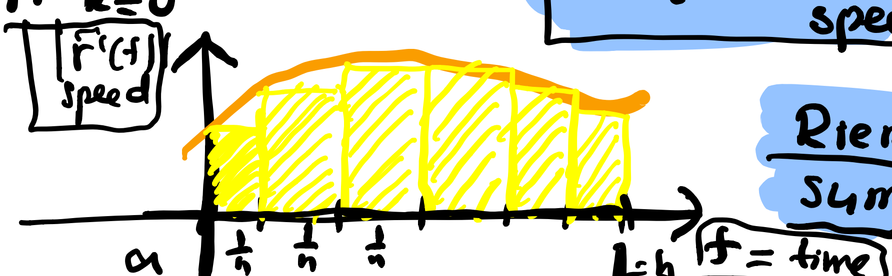
$|\vec r'(t)| \sim \left|\dfrac{\vec r(t+h) - \vec r(t)}{h}\right|$

if $h = \frac{1}{n}$

$\frac{1}{n}\,|\vec r'(t)| \sim |\vec r(t+h) - \vec r(t)|$

time · velocity ↑ displacement

$|\vec r'(t)|$ speed



Riemann sum

If $n \to \infty$, we get by definition the area under the curve

$$\boxed{L = \int_a^b |\vec{r}\,'(t)|\, dt}$$ Arc length formula

Remark! | The arc length is independent of the parametrization.

Substitution $\longrightarrow$

This is a definite integral

$$\int_a^b f(x)\,dx$$

There is no constant

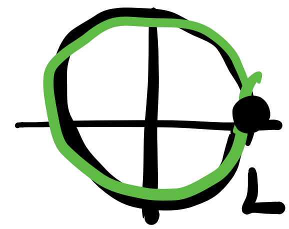
$$\int_a^b \sqrt{x'(t)^2 + y'(t)^2}\; dt$$

$$\vec{r}\,'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}, \quad |\vec{r}\,'(t)| = \sqrt{x'(t)^2 + y'(t)^2}$$

2 Examples

Circumfer of circle

a) $\vec{r}(t) = \begin{bmatrix} L\cos t \\ L\sin t \end{bmatrix}$



$\vec{r}\,'(t) = \begin{bmatrix} -L\sin t \\ L\cos t \end{bmatrix}$

$|\vec{r}\,'(t)| = \sqrt{L^2\sin^2 t + L^2\cos^2 t}$
$= L$

$\int_0^{2\pi} L\,dt = Lt\Big|_0^{2\pi} = \boxed{L \cdot 2\pi}$

Archimedes

b) $\vec{r}(t) = \begin{bmatrix} \sqrt{2}\,t \\ \frac{t^2}{2} \\ \log t \end{bmatrix}$ $\quad 1 \leq t \leq 2$

$\log(t) = \ln(t)$

Find the arc length!

$\vec{r}\,'(t) = \begin{bmatrix} \sqrt{2} \\ t \\ \frac{1}{t} \end{bmatrix}$, $|\vec{r}\,'(t)| = \sqrt{2 + t^2 + \frac{1}{t^2}}$

$$\int_1^2 \sqrt{2+t^2+\frac{1}{t^2}}\,dt$$

$\boxed{\text{Simplify! perfect square}}$

$$= \int_1^2 \sqrt{\left(t+\frac{1}{t}\right)^2}\,dt = \int_1^2 t+\frac{1}{t}\,dt$$

$$= \frac{t^2}{2} + \log t \Big|_1^2 = \boxed{\frac{3}{2} + \log(2)}$$

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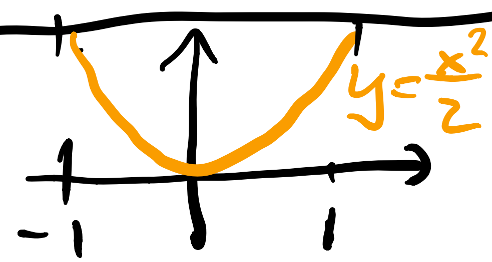
c)



What is the arc length of the parabola

$$y = \frac{x^2}{2}$$

$$-1 \le x \le 1$$

$$\vec{r}(t) = \begin{bmatrix} t \\ \frac{t^2}{2} \end{bmatrix},$$

<u>parametrize</u>

$$\vec{r}\,'(t) = \begin{bmatrix} 1 \\ t \end{bmatrix}, \quad \vec{r}\,'(t) = \sqrt{1+t^2}$$

$$\int_{-1}^{1} \sqrt{1+t^2}\,dt$$

Subst.

$u = 1+t^2$

$du = 2t\,dt$

$dt = \frac{du}{2t}$

$= \frac{du}{\ldots}$

hyperbolic trig

$t = \sinh u$

$$\int \underbrace{\sqrt{1+t^2}}_{u} \cdot \underbrace{1 \; dt}_{dV}$$

$$\sqrt{1+t^2}\cdot t - \int \frac{\cancel{2}t \cdot t}{\cancel{2}\sqrt{1+t^2}}$$

$$= \sqrt{1+t^2}\cdot t - \int \frac{t^2+1}{\sqrt{1+t^2}}\,dt + \int \frac{1}{\sqrt{1+t^2}}\,dt$$

$$\int \sqrt{1+t^2}\,dt \qquad \operatorname{arcsinh}(t)$$

$$\int \sqrt{1+t^2}\,dt = \sqrt{1+t^2}\cdot t - \int \sqrt{1+t^2}\,dt + \operatorname{arcsinh}(t)$$

$$\int \sqrt{1+t^2}\,dt = \boxed{\frac{\sqrt{1+t^2}\cdot t + \operatorname{arcsinh}(t)}{2}}$$

$$\int_{-1}^{1} \sqrt{1+t^2}\,dt = \boxed{\sqrt{2} + \operatorname{arcsinh}(1)}$$

$$\sinh(t) = \frac{e^{t}-e^{-t}}{2}$$

$\cosh' = \sinh$

$\sinh' = \cosh$

$$\cosh^2(t) - \sinh^2(t) = 1$$

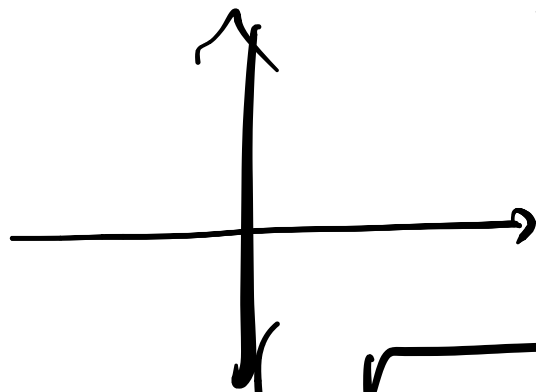
$$\boxed{\begin{aligned} \cos^2(t) &= \frac{1+\cos 2t}{2} \\ \sin^2(t) &= \frac{1-\cos 2t}{2} \end{aligned}}$$

Double angle forma's

$$\left(\frac{e^{t}+e^{-t}}{2}\right)^2 = \frac{e^{2t}+2+e^{-2t}}{4}$$

$$= \frac{\cosh(2t)+1}{2}$$

c)



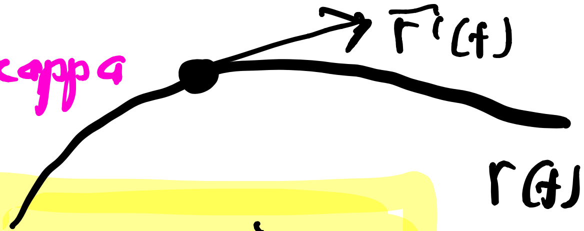
$$r(t) = \begin{bmatrix} \cos 2t \\ \sin 5t \end{bmatrix}$$

$$\sqrt{\frac{\sin^2(2t)}{4}+\frac{\cos^2(5t)}{4}}$$

In general, we can integrate this in a closed form

(4) $\boxed{\text{Curvature}}$

κ kappa



$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

unit tangent vector.

$$\boxed{\kappa = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}}$$

is the curvature of $r(t)$ at t

is independent of the parametrization

$$\boxed{\kappa = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}}$$

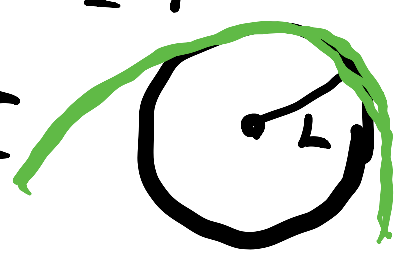
lets verify this!

$$\vec{r} = \begin{bmatrix} L\cos t \\ L\sin t \end{bmatrix}, \quad \vec{r}'(t) = \begin{bmatrix} -L\sin t \\ L\cos t \end{bmatrix}$$

$$|\vec{r}'(t)| = L, \quad \vec{T}(t) = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$$

$$T'(t) = \begin{bmatrix} -\cos t \\ -\sin t \end{bmatrix}, \quad |\vec{T}'(t)| = 1$$

$$\kappa = \frac{1}{|\vec{r}'(t)|} = \frac{1}{L}$$



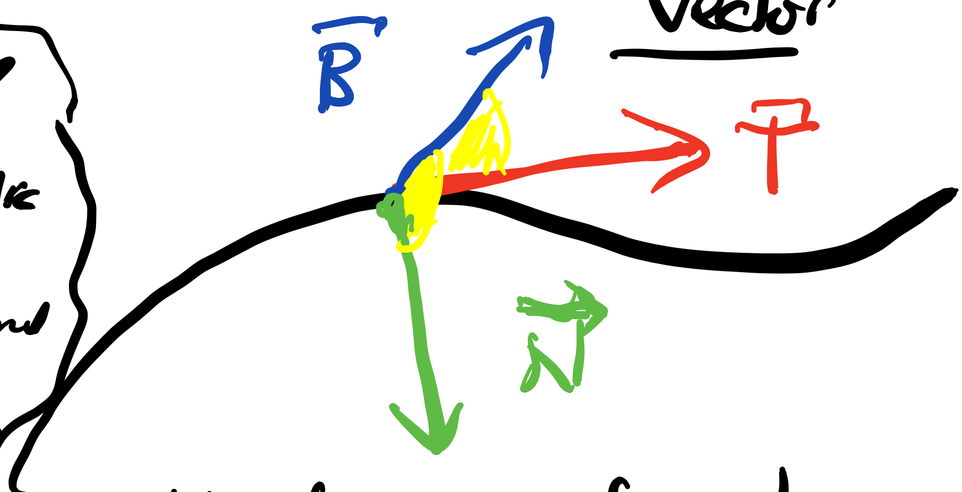
small circles have big curvature.

(5) TNB frame

$$\vec{N} = \frac{\vec{T}'}{|\vec{T}'|}$$ normal vector

$$\vec{B} = \vec{T} \times \vec{N}$$ binormal vector

Normal = perpendicular = orthogonal = uncorrelated



All these vectors have length 1 and are perpendicular to each other.

$\vec{T}$ and $\vec{N}$ are perpendicular

$$\frac{d}{dt}(\vec{T}\cdot\vec{T}) = \frac{d}{dt}1 = 0$$

$$\parallel (*)$$

$$\vec{T}'\cdot\vec{T} + \vec{T}\cdot\vec{T}' = 2\boxed{\vec{T}\cdot\vec{T}'}$$

$\Rightarrow$ $\vec{T}'$ is $\perp$ to $\vec{T}$.

$$\frac{d}{dt}\left(\vec{A}(t)\cdot\vec{B}(t)\right)$$

$$= A'(t)\,B(t) + A(t)\,B'(t)$$

Product rule

what was important, is that $\vec{T}(t)$ has constant length