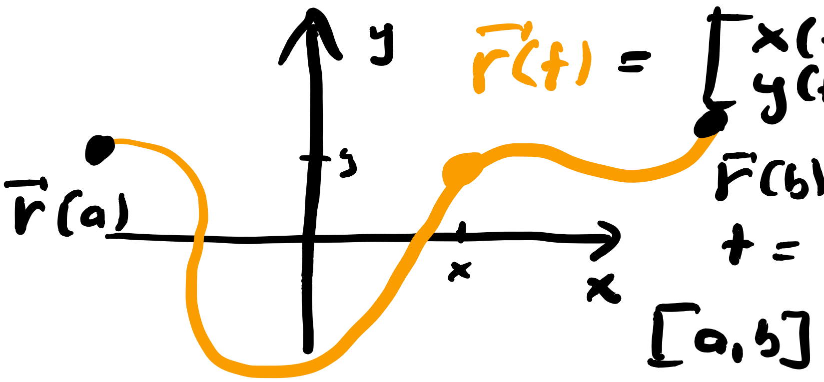
# Unit 7 Curves

## 1) Parametrization

$$\vec r(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

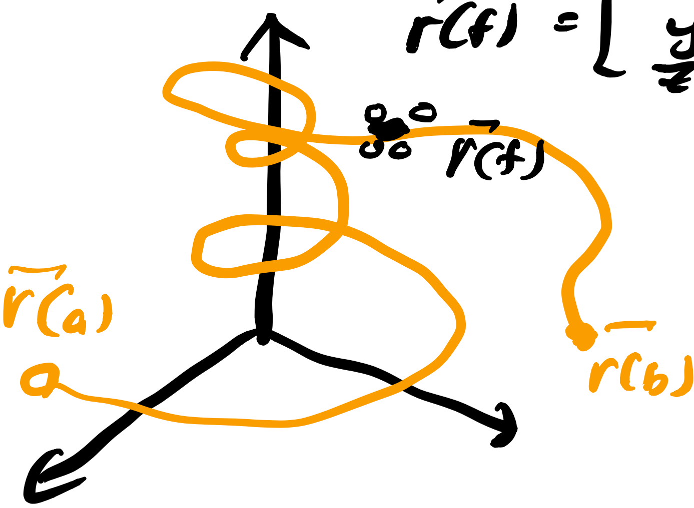



t = "time"

[a, b] time interval

In space, we have 3 coordinate

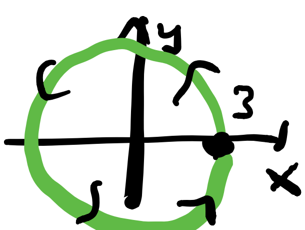
$$\vec r(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$




space curve

## 2) Examples $t \in [0, 2\pi]$

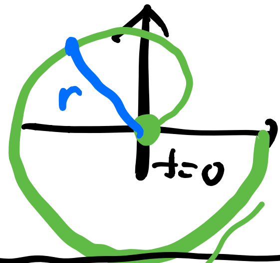
a)




circle

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 3\cos t \\ 3\sin t \end{bmatrix}$$

$$x^2 + y^2 = 9\cos^2 t + 9\sin^2 t = 9$$

b) 


$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} t\cos t \\ t\sin t \end{bmatrix}$$

r

Spiral

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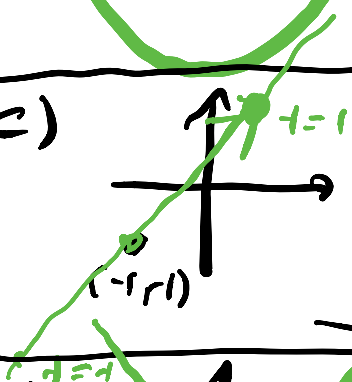
c) 


$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} + t\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} -1+3t \\ -1+4t \end{bmatrix}$$

Line

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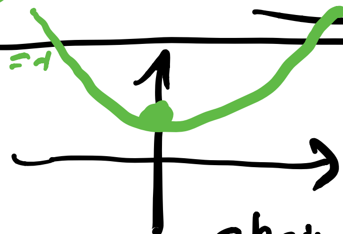
d) 

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} t \\ \frac{e^{t}+e^{-t}}{2} \end{bmatrix}$$

chain curve

$$= \begin{bmatrix} t \\ \cosh(t) \end{bmatrix}$$

hyperbolic ~~trig~~ function.

$$\frac{e^{it}+e^{-it}}{2} = \cos x \qquad e^{it} = \cos t + i\sin t$$

$$\boxed{e^{i\pi}+1=0}$$

$\pi, e, i, 1, 0$

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This is an example of a graph

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} t \\ f(t) \end{bmatrix}, \quad y = f(x)$$

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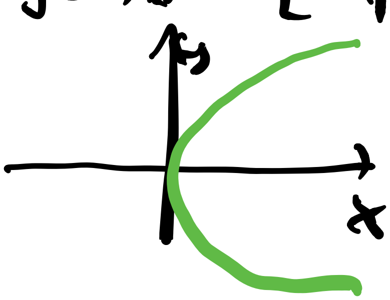
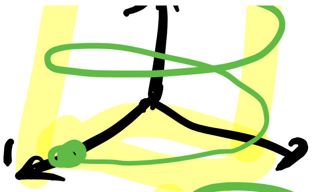
$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} f(t) \\ t \end{bmatrix} = \begin{bmatrix} t^2 \\ t \end{bmatrix}$$




$x = y^2$

$y = \pm\sqrt{x}$

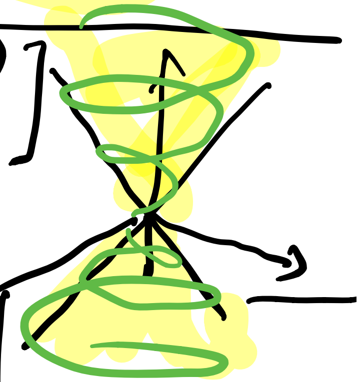
e) $\vec{r}(t) = \begin{bmatrix} \cos t \\ \sin t \\ t \end{bmatrix}$ $\quad x^2+y^2=1$



Helix.

f) $\vec{r}(t) = \begin{bmatrix} t\cos t \\ t\sin t \\ t \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$

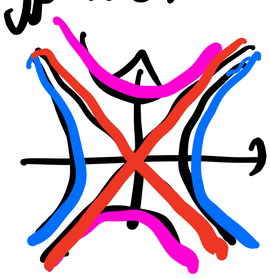
$$x^2+y^2=z^2$$



g) $\vec{r}(t) = \begin{bmatrix} \sqrt{t^2+1}\,\cos t \\ \sqrt{t^2+1}\,\sin t \\ t \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$x^2+y^2 = t^2+1 = z^2+1 \qquad \boxed{x^2+y^2-z^2=1}$$

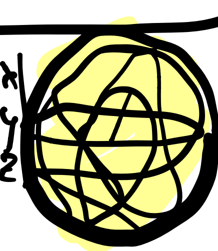
One sheeted hyperbolic



$\begin{bmatrix} \sqrt{t^2-1}\,\cos t \\ \sqrt{t^2-1}\,\sin t \\ t \end{bmatrix}$

on a 2 sheeted hyperb.

h) $\vec{r}(t) = \begin{bmatrix} \sin(3t)\cos(t\cdot 5) \\ \sin(3t)\sin(t\cdot 5) \\ \cos 3t \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$



this is a curve on a sphere because $x^2+y^2+z^2=1$


r

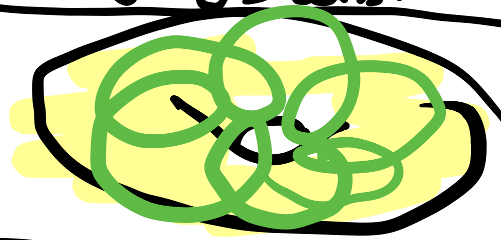
$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}\sin s \cos t\\ \sin s \sin t\\ \cos s\end{bmatrix}$$ Sphere

For curves, we have only one parameter
For surfaces, we have two parameters

i) grid curves:

grid curve
$\theta = \text{const}$

HW:

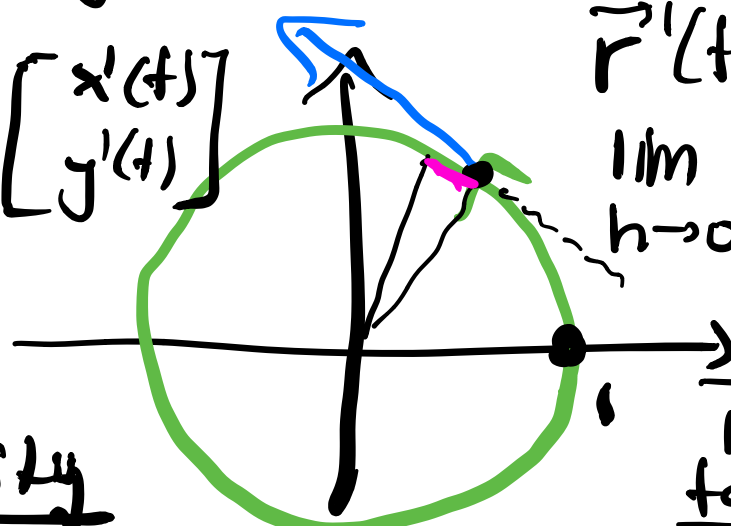


Parallelepiped

③ Differentiation

$$\vec{r}(t) = \begin{bmatrix}x(t)\\y(t)\end{bmatrix} = \begin{bmatrix}\cos t\\ \sin t\end{bmatrix}$$

$$\vec{r}\,'(t) = \begin{bmatrix}x'(t)\\y'(t)\end{bmatrix}$$




$$\vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

Velocity

$\vec{r}\,'(t)$ is tangent to the curve

$$\vec{r}''(t) = \begin{bmatrix} x''(t) \\ y''(t) \end{bmatrix}$$ acceleration

$$\boxed{m\,\vec{r}''(t) = \text{Force}}$$ $\boxed{\text{Newton}}$

buttle

$$\vec{r}'''(t) = \begin{bmatrix} x'''(t) \\ y'''(t) \end{bmatrix} = \text{jerk}$$

$\vec{r}''''(t)$ = snap, $\vec{r}^{(5)}$ = crackle

$\vec{r}^{(6)}$ = pop, $\vec{r}^{(7)}$ = Harvard

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whistle.

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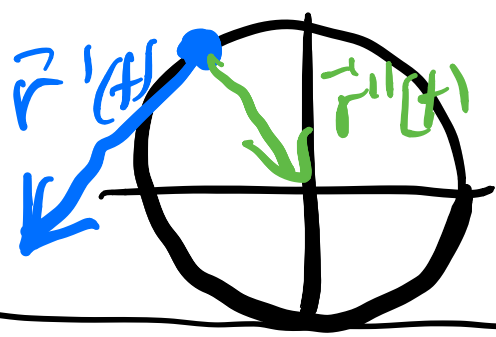
$$\vec{r}'(t) = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}, \quad \vec{r}''(t) = \begin{bmatrix} -\cos t \\ -\sin t \end{bmatrix}$$

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$$\vec{r}(t) = \begin{bmatrix} \sin 2t \\ \cos 2t \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

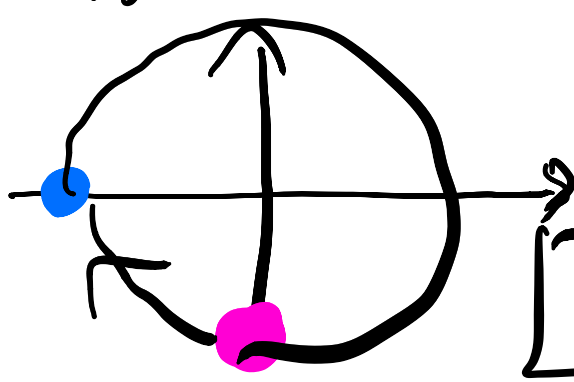
is still a circle, but it is traversed fast $\qquad x^2+y^2=1$

$$\vec{r}'(t) = \begin{bmatrix} +2\cos 2t \\ -2\sin 2t \end{bmatrix}$$

The curve is not the same than the parametrization the later contains much inform. like how fast we are going.

clockwise rotation $\begin{bmatrix} -\sin 2t \\ \cos 2t \end{bmatrix}$

$\boxed{\text{TF}}$ $\vec{r}(t) = \begin{bmatrix} -\sin t \\ -\cos t \end{bmatrix}$ is a circle <u>rotating</u> counterclockwise.



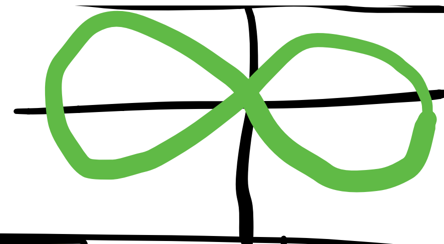
$t=0$

$t=\frac{\pi}{2}$

$\boxed{\text{False}}$

What is this? $\boxed{0.9999999\ldots = 1}$

$$\begin{bmatrix}\sin 2t\\ \cos t\end{bmatrix}$$



## 4 Integrations

FTC in vector form


$$\vec{v}(t) = \begin{bmatrix} x(t)\\ y(t)\end{bmatrix}$$

$$\boxed{\vec{r}(t) = \int_0^t \vec{v}(s)\,ds + \vec{r}(0)}$$

has the property that

$$\vec{r}\,'(t) = \vec{v}(t)$$

$\vec{v}(t) =$

$$\vec{r}\,'(t) = \begin{bmatrix}\sin 2t\\ \cos t\end{bmatrix}, \quad \int_0^t \vec{r}\,'(s)\,ds$$

$$= \int_0^t \begin{bmatrix}\sin 2s\\ \cos s\end{bmatrix} ds = \begin{bmatrix}-\frac{1}{2}\cos 2s\\ \sin s\end{bmatrix}_0^t$$

Fixed so that $\vec{r}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\longrightarrow$ $= \begin{bmatrix} \frac{1}{2} - \frac{1}{2}\cos 2t \\ \sin t \end{bmatrix}$

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## (5) Free Fall

$\vec{a}(t)$ acceleration

$\vec{r}''(t)$ is given.

fix the const

$$\boxed{\vec{r}'(t) = \int_0^t \vec{r}''(s)\,ds + \vec{r}'(0)}$$

$$\boxed{\vec{r}(t) = \int_0^t \vec{r}'(s)\,ds + \vec{r}(0)}$$

fix the const

**Example!**

Assume

$\vec{r}''(t) = \begin{bmatrix} \cos t + t \\ \sin t \end{bmatrix}$

$\vec{r}'(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \vec{r}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Find $\vec{r}(t)$!

Epicycle

First step

$$\left[\begin{aligned} \vec{r}''(t) &= \begin{bmatrix} \cos t + t \\ \sin t \end{bmatrix} \\ \vec{r}'(t) &= \begin{bmatrix} \sin t + \frac{t^2}{2} \\ -\cos t \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \sin t + \frac{t^2}{2} \\ 1 - \cos t \end{bmatrix} \end{aligned}\right.$$

$$\vec{r}(t) = \begin{bmatrix} -\cos t + \frac{t^3}{6} \\ t - \sin t \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

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Where does the $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$ come from?

Put $t = 0$ : $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$

should be $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

we have to add $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$

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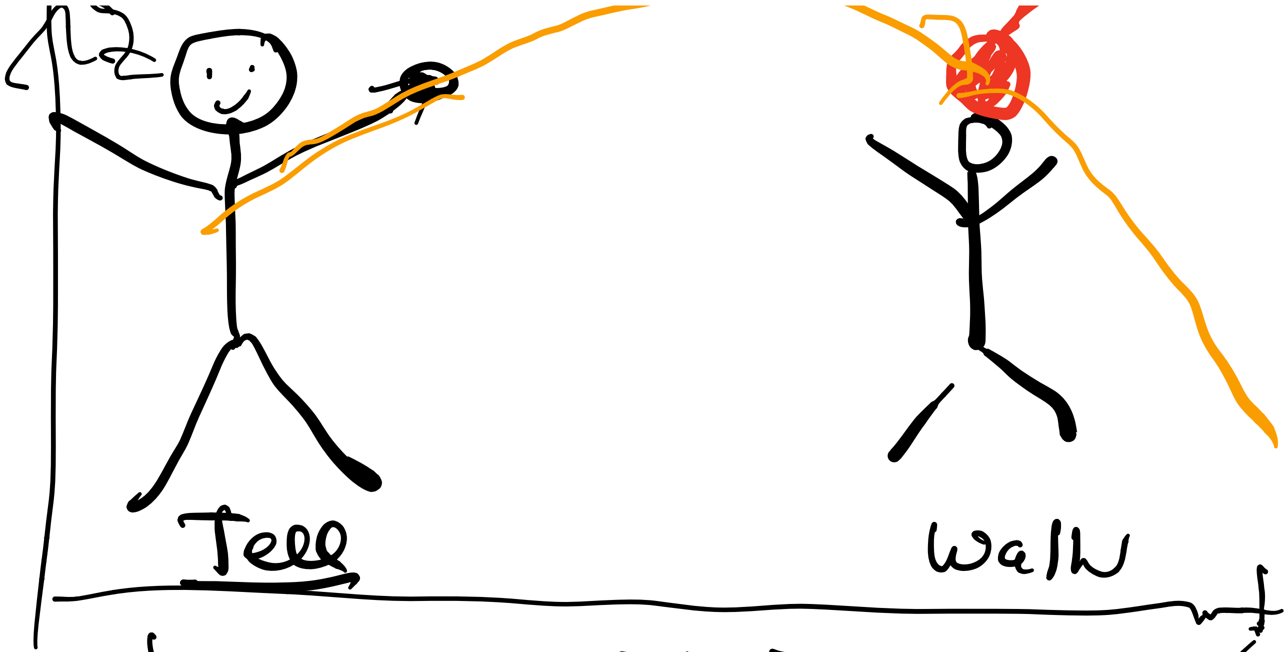
Wilhelm Tell

f




$$\vec{r}\,'(0) = \begin{bmatrix} 10 \\ 0 \\ 2 \end{bmatrix}$$ is given

$$\vec{r}(0) = \begin{bmatrix} 1 \\ 0 \\ 6 \end{bmatrix}$$

$$\vec{r}\,''(t) = \begin{bmatrix} 0 \\ 0 \\ -10 \end{bmatrix}$$ gravitation force

Find $\vec{r}(t)$ !

$$\vec{r}''(t) = \begin{bmatrix} 0 \\ 0 \\ -10 \end{bmatrix}$$

$$\vec{r}'(t) = \begin{bmatrix} 0 \\ 0 \\ -10t \end{bmatrix} + \begin{bmatrix} 10 \\ 0 \\ 2 \end{bmatrix}$$

velocity of arrow $= \begin{bmatrix} 10 \\ 0 \\ 2-10t \end{bmatrix}$

$$\vec{r}(t) = \begin{bmatrix} 10t \\ 0 \\ 2t-5t^2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 6 \end{bmatrix}$$

$$\boxed{\vec{r}(t) = \begin{bmatrix} 1+10t \\ 0 \\ 6+2t-5t^2 \end{bmatrix}}$$

What curve is this?

Parabola.

$z =$ quadratic in $x$

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$6 + 2t - 5t^2 = 0$

$\rightarrow$ time when the arrow hits the ground

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$t = \dfrac{x-1}{10}$

$z = 6 + 2\left(\dfrac{x-1}{10}\right) - 5\left(\dfrac{x-1}{10}\right)^2$

$= -\dfrac{5}{100}x^2 + ax + b$

This is a parabola in the $xz$ plane.

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Tom Petty:

Free Falling

Find an antiderivative of
$\sin x$

$-\cos x + C$

Find the antiderivative
such that $\boxed{f(0) = 5}$

$-\cos x + 6$