8/6/2020 PRACTICE EXAM III **Maths 21a, O. Knill, Summer 2020**

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) (20 points) No justifications are necessary

1) T F — For any vector field $\vec{F} = [P, Q, R]$, the identity $\text{div}(\text{curl}(\vec{F})) = |\text{grad}(\text{div}(\vec{F}))|$ holds.

Solution:
The first is zero, the second not necessarily.

2) T F — The arc length of any curve C contained on the unit sphere $x^2 + y^2 + z^2 = 1$ is smaller or equal than 2π.

Solution:
This is true for circles but not for general curves.

3) T F — The integral $\iint_R 1 \; dudv$ is the surface area of a surface parametrized by $R \to S, (u, v) \to \vec{r}(u, v)$.

Solution:
The integration factor $|\vec{r}_u \times \vec{r}_v|$ is missing.

4) T F — The equation $\text{div}(\text{grad}(f) = 0$ is an example of a partial differential equation.

Solution:
yes, it is an equation for an unknown function f involving derivatives with respect at least different variables.

5) T F — A mass point moving on a curve $\vec{r}(t)$ for which the acceleration $\vec{r}\,''(t)$ is zero for all t remains either at a point or moves on a straight line.

Solution:
This is one of Newton's law.

6) T F — Given a closed surface S and a constant vector field $\vec{F} = [2, 3, 4]$, then the flux of $\vec{F}$ through S is zero.

Solution:
Indeed, the vector field is a curl of an other vector field.

7) | T | F | A vector field $\vec{F}$ which is incompressible is by Clairaut also irrotational.

Solution:
Irrotational and incompressible have no relations.

8) | T | F | The curvature of any point in the ellipse $x^2/4+y^2=1$ is everywhere smaller than 1.

Solution:
It is larger on the on the x axes.

9) | T | F | Two curves are parametrized by $\vec{r}(t), \vec{s}(t)$. As they intersect at a point P, there exists a time t such that they collide $\vec{r}(t)=\vec{s}(t)$.

Solution:
We have seen that particle paths can intersect without the particles need to collide.

10) | T | F | Given three curves L, M, K. If $d(L,M), d(M,K), d(L,K)$ are the distances between the curves, then the triangle inequality $d(L,M)+d(M,K) \geq d(L,K)$ holds.

Solution:
Take three circles $L=\{(x-1.5)^2+y^2=1\}$, $M=\{x^2+y^2=1\}$, $K=\{(x+1.5)^2+y^2=1\}$. Then $d(L,M)=d(M,K)=0$ but $d(L,K)>0$.

11) | T | F | If $\vec{F}$ and $\vec{G}$ are two vector fields for which the divergence is the same. Then $\vec{F}-\vec{G}$ is a constant vector field.

Solution:
Take for example $\vec{F}=[3x^2-y^2,0,0]$ and $\vec{G}=[0,6xy,0]$. They both have divergence $6x$ but their difference is not constant.

12) T **F**

Let S is the unit sphere, oriented outwards and $\vec{F}$ is a vector field in space which is a gradient field then $\iint_S \vec{F} \cdot d\vec{S} = 0$.

Solution:
We would need a curve and line integral not a surface and flux integral.

13) **T** F

If $\vec{F}, \vec{G}$ are two vector fields which have the same curl, then $\vec{F} - \vec{G}$ is a gradient field.

Solution:
Yes, this holds in general if the region is simplify connected.

14) **T** F

The expression div(grad(div(curl(curl(grad(f)))))) is a well defined function of three variables if f is a function of three variables.

Solution:
Every of the expressions make sense. Not well defined would be grad(grad(f)) for example as the gradient is only defined for scalar functions.

15) T **F**

The parametrization $\vec{r}(u, v) = [u + v, u + v, u + v]$ describes a plane.

Solution:
It is a line

16) **T** F

Any function $u(x, y)$ that obeys the differential equation $u_{xx} + u_y = 1$ has no local maxima.

Solution:
At a local maximum, we have $\nabla u = [u_x, u_y] = [0, 0]$, so that $u_{xx} = 1$ which is incompatible with a local maximum, where $u_{xx} < 0$ by the second derivative test.

17) **T** F

If $\vec{F}(x, y, z)$ is a vector field so that $[P_x, Q_y, R_z] = [0, 0, 0]$, then it is incompressible.

Solution:
Indeed, $P_x + Q_y + R_z$ is then zero.

18) | T | **F** (highlighted) |

If $f(x,y)=0$ is a level set and $f(x,g(y))=0$, then $g_x = -f_x/f_y$ provided f_y is not zero.

Solution:
This would be implicit differentiation if $f(x,g(x)) = 0$ but as it stands, we can take $g(y) = y$.

19) | **T** (highlighted) | F |

By linear approximation we can estimate $\sqrt{10001} = 100 + 1/200$.

Solution:
Yes, this is one-dimensional linear approximation.

20) | **T** (highlighted) | F |

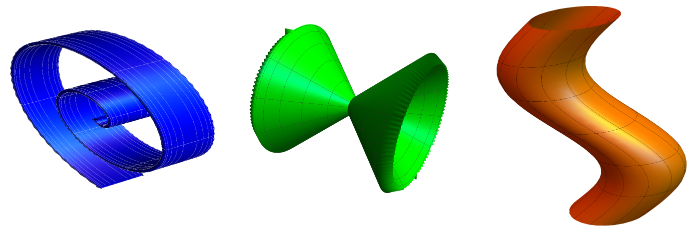
The flux of $\vec{F}(\vec{r}(u,v)) = (\vec{r}_u \times \vec{r}_v)/|\vec{r}_u \times \vec{r}_v|$ through an ellipsoid surface S parametrized by $\vec{r}(u,v)$ is the surface area of S.

Solution:
This was a homework problem.

Problem 2) (10 points) No justifications are necessary.

a) (2 points) Match the following surfaces. There is an exact match.

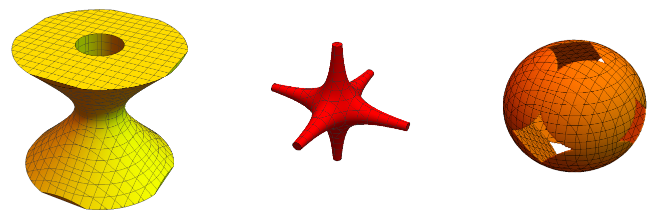
A B C



Parametrized surface $\vec{r}(u,v)$	A-C
$[\sin(v)+\sin(u), \cos(u), u]$	
$[v, v\sin(u), v\cos(u)]$	
$[v\sin(v), v\cos(v), \sin(u)]$	

b) (2 points) Match the solids. There is an exact match.

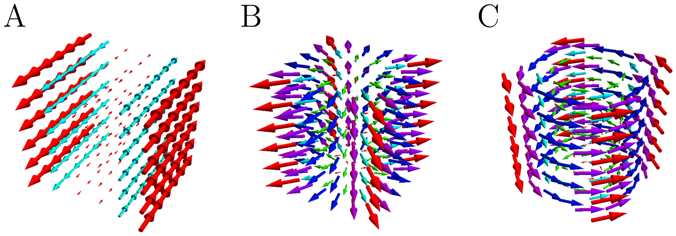
A B C



Solid	A-C
$x^2y^2+x^2z^2+y^2z^2<1$	
$x^2+y^2>1, x^2+y^2-z^2<1$	
$\lvert(x,y,z)\rvert<3, \lvert x\rvert+\lvert y\rvert+\lvert z\rvert>3$	

c) (2 points) The figures display vector fields. There is an exact match.

A B C



Field	A-C
$\vec{F}(x,y,z)=[x,y,0]$	
$\vec{F}(x,y,z)=[-y,x,0]$	
$\vec{F}(x,y,z)=[0,x,0]$	

d) (2 points) Recognize partial differential equations!

Equation	A-F
Wave	
Transport	
Burger	

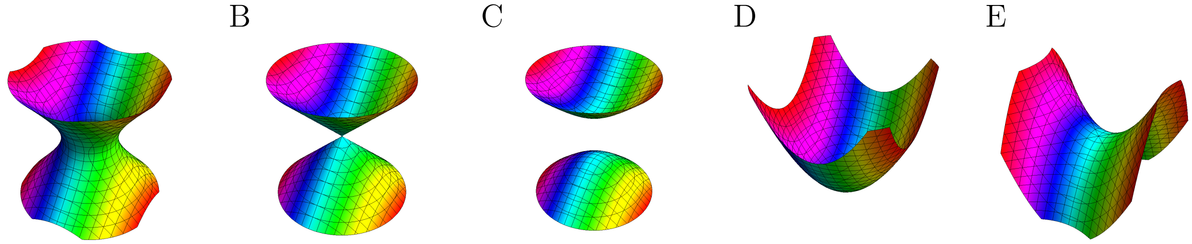
	PDE
A	$f_t+f_x=0$
B	$f_{tt}-f_{xx}=0$
C	$f_t+ff_x=0$

	PDE
D	$f_t-f_x=0$
E	$f_{tt}+f_{xx}=0$
F	$f_t^2+f_x^2=1$

e) (2 points) Find the quadrics!

	Enter A-E
Which one is the one sheeted hyperboloid?	
Which one is the hyperbolic parabolid?	

A B C D E



Solution:
a) CBA
b) BAC
c) BCA
d) BAD or BAC
e) AE

Problem 3) (10 points) No justifications necessary

a) (5 points) Complete the **signs**. Fill in either an **equal sign** (=) a **less or equal sign** ($\leq$) or **larger than equal sign** ($\geq$). Each correct answer is a point.

Formula		
$\vec{v} \cdot \vec{w}$		$\lvert\vec{v}\rvert\lvert\vec{w}\rvert\cos(\theta)$
$\vec{v} \cdot \vec{w}$		$\lvert\vec{v}\rvert\lvert\vec{w}\rvert$
$\lvert\vec{v}\rvert + \lvert\vec{w}\rvert$		$\lvert\vec{v} + \vec{w}\rvert$
$\int_0^1 \lvert\vec{r}\,'(t)\rvert\, dt$		$\lvert\int_0^1 \vec{r}\,'(t)\, dt\rvert$
$\lvert\vec{v} \times \vec{w}\rvert$		$\lvert\vec{v}\rvert\lvert\vec{w}\rvert$

We have seen several fundamental theorems. We want you to write down the results.

b) (1 point) What is the fundamental theorem of gradients?

c) (1 point) What is the fundamental theorem of line integrals?

d) (1 point) What is Clairaut's theorem?

e) (1 point) What is Fubini's result?

f) (1 point) Write down a relation between curl, grad and div which is always zero.

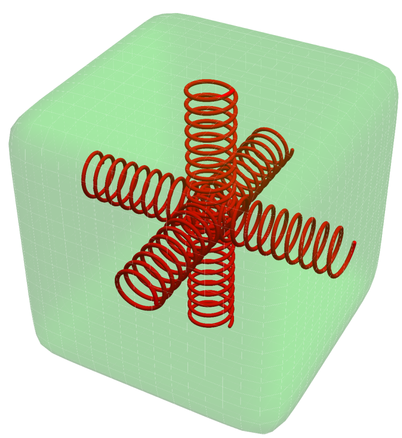
Solution:
a) $=, \leq, \geq, \geq$ and $\leq$.
b) The gradient $\nabla f(x)$ is perpendicular to level sets $f(x) = c$.
c) $\int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt = f(\vec{r}(b)) - f(\vec{r}(a))$.
d) $f_{xy} = f_{yx}$.
e) $\int_a^b \int_c^d f(x, y)\; dydx = \int_c^d \int_a^b f(x, y)\; dxdy$.
f) $\text{curl}(\text{grad}(f)) = 0$ or $\text{div}(\text{curl}(\vec{F})) = 0$.

Problem 4) (10 points)

A modern design shows a lamp in which three carbon springs hold together a thin silk hull. We advertise it as **"new materials and natural fabric in a symbiotic equilibrium".** Assume the velocity of the curve is given by

$$\vec{r}\,'(t) = [-10\sin(10t), 10\cos(10t), 1]$$

and that $\vec{r}(0) = [0, 10, 0]$.



a) (2 points) What is the unit tangent vector $\vec{T}(0)$ at $t = 0$?

b) (4 points) Find the parametrization $\vec{r}(t)$ of the curve.

c) (4 points) Find the total arc length of this curve if $-2\pi \leq t \leq 2\pi$.

Solution:

a) $\vec{r}\,'(0) = [0, 10, 1]$. Now normalize to $\vec{T}\,'(0) = \boxed{[0, 10/\sqrt{101}, 1/\sqrt{101}]}$.

b) Integrate to get

$$\vec{r}(t) = [\cos(10t), \sin(10t), t] + [C_1, C_2, C_3]$$

Fix the constants to get it right at $t = 0$:

$$\vec{r}(t) = [\cos(10t) - 1, \sin(10t) + 10, t]\ .$$

c) $\int_{-2\pi}^{2\pi} |[-10\sin(10t), 10\cos(10t), 1]|\ dt = \int_{-2\pi}^{2\pi} \sqrt{101}\ dt = 4\pi\sqrt{101}$. Thie final result is $\boxed{4\pi\sqrt{101}}$.

Problem 5) (10 points)

a) (3 points) A new **t-shirt** shows the curve

$$x^2 + (y - x^{2/3})^2 = 1\ .$$

What is the tangent line at the point $(1, 1)$?

b) (3 points) Estimate $1.001^2 + (1.01 - 1.001^{2/3})^2$ using linear approximation.

c) (4 points) A 3D implementation of the heart is the surface

$$x^2 + (y - x^{2/3})^2 + 16z^2 = 1\ .$$

The point $(1, 1, 0)$ is on this surface. What is the tangent plane to the surface at this point?

Solution:
a) The gradient of the equation $f(x,y) = 1$ is

$$[a,b] = [2x, 2(y - x^{2/3})(2/3)x^{-1/3}]$$

which is $[2,0]$ at the point we are interested in. The equation of the line is therefore $ax + by = d$ where d is the constant obtained by plugging in the point. Here $2x = 2$ so that $\boxed{x = 1}$ is the line.

b) We use the linearization $L(x,y) = 1 + 2(1.001 - 1) = \boxed{1.002}$.

c) The gradient of the equation $g(x,y,z) = 1$ is

$$[2x, 2(y - x^{2/3})(2/3)x^{-1/3}, 32z]$$

At the point $(1,1,0)$, this is

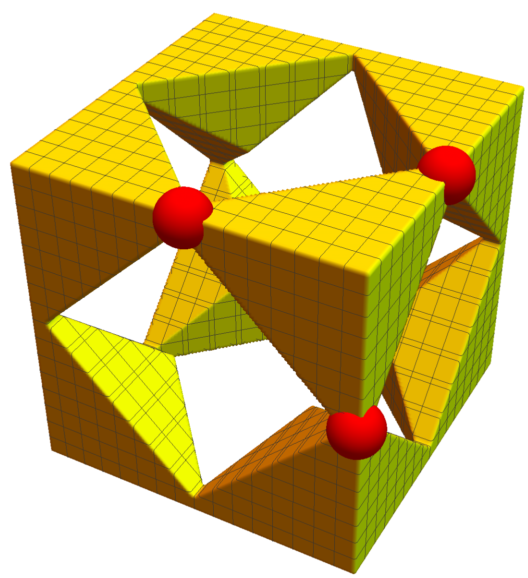
$$[a,b,c] = [2,0,0] \ .$$

The equation of the plane is $ax+by+cz = d$ where the constant d is obtained by plugging in the point $(1,1,0)$. The result is $2x = 2$ or $\boxed{x = 1}$.

P.S. In the actual exam, the simpler equation $x^2+(y-x^{1/3})^2 = 1$ was used which produces a very similar heart and is a bit simpler but there were questions during the exam asking why that equation does not match the t-shirt. Therefore, in this document it has been changed to match the t-shirt. None of the final results are affected.

Problem 6) (10 points)

The **connection cube** is the complement of the truncated cube within the cube. Having realized it as a pasta, we cut it along the plane containing the points $A = (1,1,0), B = (1,0,1)$ and $C = (0,1,1)$.



a) (3 points) Find the equation of that plane.

b) (3 points) What is the area of the triangle ABC?

c) (4 points) Find the distance of that plane to the center $(0,0,0)$.

Solution:

a) The equation of the plane is obtained by finding the normal vector $\vec{n} = \vec{AB} \times \vec{AC} = [0, -1, 1] \cdot [-1, 0, 1] = [-1, -1, -1]$. The equation of the plane is $x + y + z = d$. The constant d can be obtained by plugging in one point so that $\boxed{x + y + z = 2}$.

b) The area is half the length of the vector $\vec{n}$ we just computed. It is $\boxed{\sqrt{3}/2}$.

c) The distance is given by the distance formula $\vec{OA} \cdot \vec{n}/|\vec{n}|$ which is a volume divided by the base area. as the dot product is 2, we get $\boxed{2/\sqrt{3}}$.

Problem 7) (10 points)

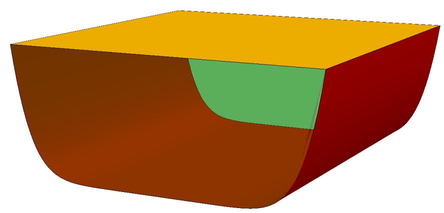
A Boston **duck tour boat** is bound from below by the function

$$f(x, y) = x^8$$

and from the top by the function

$$g(x, y) = 1 \ .$$

It is also bound by the planes $y = -1$ and $y = 1$.



a) (6 points) Find the volume of the boat.

b) (4 points) Write down the double integral which gives the surface area of the bottom part, the graph of the function f above the square. You don't have to evaluate the integral.

Solution:
a) The boat is sandwiched between two graphs defined over the square $[-1,1] \times [-1,1]$. The reason is that the graphs $y = x^8$ and $y = 1$ intersect on the lines $x = -1$ or $x = 1$. The integral is

$$\int_{-1}^{1} \int_{-1}^{1} \int_{x^8}^{1} 1 \, dzdydx \, .$$

Start solving this from the inner integral on to get

$$\int_{-1}^{1} \int_{-1}^{1} (1 - x^8) \, dydx \, .$$

Then the next integral

$$\int_{-1}^{1} 2 - 2x^8 \, dx$$

which finally is $4 - 4/9 = \boxed{32/9}$.
b) To get the surface area of the graph, we need to parametrize the graph first. This is a graph so that we have

$$\vec{r}(x,y) = [x, y, x^8] \, .$$

Now we use the formula $\int \int_R |r_x \times r_y| \, dydx$. This is

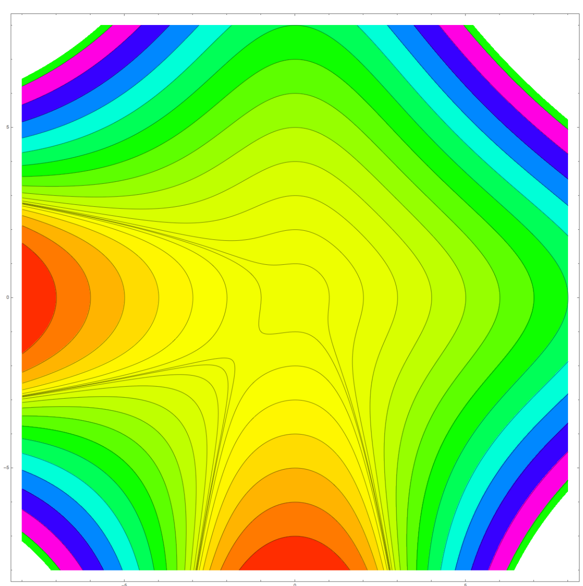
$$\int_{-1}^{1} \int_{-1}^{1} |[1, 0, 8x^7] \times [0, 1, 0]| \, dydx$$

We can still simplify to

$$2 \int_{-1}^{1} \sqrt{1 + 64x^{14}} \, dx \, .$$

P.S. This integral can not be evaluated without using special functions. The numerical answer is about 3.44.

Problem 8) (10 points)



The **Laplacian** of a function $g(x,y)$ is defined as

$$\Delta g = g_{xx} + g_{yy} \, .$$

Let $g(x,y) = x^2y^2 + x^3 + y^3$.

a) (3 points) Write down the function $f(x,y)$ which is the Laplacian of g.

b) (7 points) Classify all critical points of f using the second derivative test.

Solution:
a) The function $f(x, y)$ is $6x + 6y + 2x^2 + 2y^2$.
b) We are looking for critical points $[6+4x, 6+4y] = [0, 0]$ so that $\boxed{(x, y) = (-3/2, -3/2)}$ is the only critical point. We also compute $f_{xx} = 4$ and the discriminant $D = f_{xx}f_{yy} - f_{xy}^2 = 16$. The second derivative test shows that we have found a minimum.

Problem 9) (10 points)

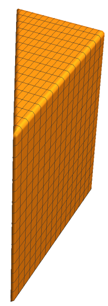
When you eat **raclette**, a swiss speciality, you place the cheese near the open fire. The part exposed to the fire melts. You scape that off and eat it with potatos, pearl onions ("petit poireau antillais") and pickels ("cornishons") and white wine. Our **raclette cheese** has volume

$$f(x, y) = x^2y/2$$

and fixed surface area

$$g(x, y) = x^2 + xy = 3$$

exposed to the fire. Using Lagrange, find the cheese parameters x, y which has maximal volume.



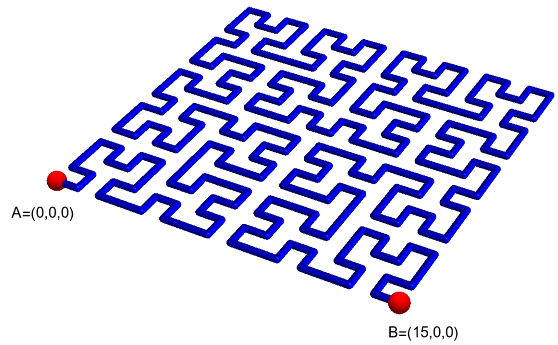
Solution:
The Lagrange equations are

$$\begin{aligned} xy &= \lambda 2x + y \\ x^2/2 &= \lambda x \\ x^2 + xy &= 3 \end{aligned}$$

Eliminating λ from the first two equations gives $2y/x = 2 + y/x$ or $y = 2x$. Plugging this into the third equation gives $x = \pm 1$ and $y = 2x$ so that $(1, 2)$, $(-1, -2)$ are solutions to the Lagrange equation. Since cheese has never negative dimension, the point $\boxed{(1, 2)}$ is the point we are looking for. It is the maximum.

Problem 10) (10 points)

Peano curves and **Hilbert curves** are famous in topology. The allow for example to prove that there is a continuous map from the interval to the square which reaches every point of the square!



Lets look at the Hilbert curve $C = H(4)$ displayed in the picture. The curve is parametrized by $\vec{r}(t)$. It starts at $A = (0,0,0) = \vec{r}(0)$ and ends at $B = (15,0,0) = \vec{r}(1)$.
What is the line integral $\int_C \vec{F} \cdot \vec{dr}$ of the vector field

$$\vec{F}(x,y,z) = [x^4 + 2xy^2z^2, y^4 + 2x^2yz^2, z^4 + 2x^2y^2z]$$

along the curve C?

P.S. A comedian once defined "Mathematics as the science where you prove things which are obvious or things which are obviously false." An example of the later is the continuous surjection from the interval and the square.

Solution:
Because the curl of $\vec{F}$ is zero, the vector field is a gradient field. We can use the fundamental theorem of line integrals. To do so, we need to find a potential $f(x,y,z)$. Integration of $P = f_x$ gives $f(x,y,z) = x^5/5 + x^2y^2z^2 + C(y,z)$. Now differentiate that with respect to y and z to get $C_y = y^5$ and $C_z = z^5$ which has the solution $C(y,z) = y^5/5 + z^5/5$. The potential is

$$f(x,y,z) = x^5/5 + y^5/5 + z^5/5 + x^2y^2z^2 .$$

The line integral is therefore $f(15,0,0) - f(0,0,0) = 15^5/5 = 3 \cdot 15^4$. The result is $\boxed{15^4}$.

Problem 11) (10 points)

At Porter square in Cambridge, at the intersection of Roseland Street and Mass Ave, a cool **bike locking arc** has recently been installed. Mathematically it represents a "link".
We place one of the arcs into the plane and model it with the curve

$$\vec{r}(t) = \begin{cases} \left[\ 2\cos(t), \sin(t) - \sin^2(t)\ \right], & 0 \leq t \leq \pi \\ \left[\ 2\cos(t), \sin(t)\ \right], & \pi \leq t \leq 2\pi \end{cases}$$

Find the area of the two-dimensional region G enclosed by this curve.

Solution:
This is a problem for Greens theorem. Take the vector field $\vec{F} = [0, x]$ which has curl 1 and compute the line integral $\int_C \vec{F} \cdot \vec{dr}$. It is the sum of two integrals:

$$\int_0^\pi [0, 2\cos(t)] \cdot [\cdot[-2\sin(t), \cos(t) - 2\sin(t)\cos(t)] = \pi - 8/6$$

and

$$\int_\pi^{2\pi} [0, 2\cos(t)] \cdot [\cdot[-2\sin(t), \cos(t)] = \pi$$

The result is $\boxed{2 - 8/6}$.

Problem 12) (10 points)

A **model train** can be built from a copper coil in which a battery and magnet moves. A problem related to that is to find the flux of the curl of the magnetic vector field

$$\vec{F}(x, y, z) = [-y, x, 0]$$

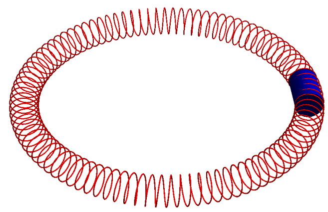
through the surface parametrized by

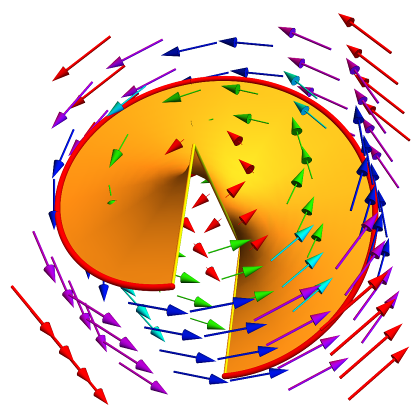
$$S : \vec{r}(s, t) = [s\cos(t), s\sin(t), t] ,$$

where $0 \leq t \leq 2\pi, 0 \leq s \leq 1$. The surface is identified on the top and bottom so the **boundary curve of the surface** is

$$C : \vec{r}(1, t) = [\cos(t), \sin(t), t] , 0 \leq t \leq 2\pi .$$

You are required to use an integral theorem. No credit for direct flux computations will be given.





The top and bottom are identified so that the boundary consists of the thickened curve only. The three straight lines which additionally appear to bound the surface are ignored. We don't make a mistake by ignoring them as their line integrals contribute nothing.

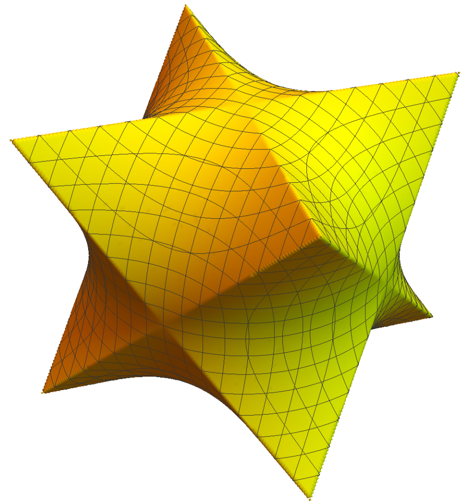
Solution:
This is a problem for Stokes theorem $\iint_S \vec{F} \cdot dS = \int_C \vec{F} \cdot \vec{dr}$. We just have to find the line integral along the boundary curve

$$\int_0^{2\pi} [-\sin(t), \cos(t), 0] \cdot [-\sin(t), \cos(t), 0] \; dr = \int_0^{2\pi} 1 \; dt = 2\pi .$$

Problem 13) (10 points)

Archimedes computed the volume of the intersection of three cylinders. The **Archimedes Revenge** is the problem to determine the volume V of the solid R defined by

$$x^2+y^2-z^2 \leq 1, y^2+z^2-x^2 \leq 1, z^2+x^2-y^2 \leq 1 .$$



Archimedes Revenge is brutal! It is definitely to hard for this exam. We give you therefore the volume $V = \log(256)$. Now to the **actual exam problem**: find the flux

$$\iint_S \vec{F} \cdot d\vec{S}$$

of

$$\vec{F}(x,y,z) = [2x+y^2+z^2, x^2+2y+z^2, x^2+y^2+2z]$$

through the boundary surface S of R, assuming that S is oriented outwards.

If you have been part of this Summer 2017 course and send me a handwritten correct solution of the **Archimedes Revenge** problem until December 31, 2017, you will earn a small prize (gift card) and admiration. The rules are that you have to solve the problem on your own and that no part of the solution should rely on computer algebra systems.

Solution:
This is a problem for the divergence theorem.

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \text{div}(\vec{V}) \, dV .$$

As the divergence is constant 6, the result will be 6 times the volume of the solid. It is $\iiint_E 6dV = 6$ Volume(E) $= \boxed{6\log(256)}$.

Problem 14) (10 points)

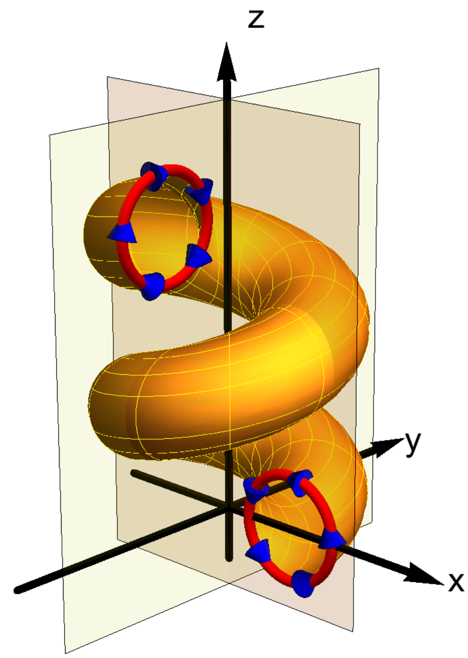
Find the **flux** of the curl of the vector field

$$\vec{F}(x,y,z) = [-z, z+\sin(xyz), x-3] + [x^5, y^7, z^4]$$

through the **twisted surface** oriented inwards and parametrized by

$$\vec{r}(t,s) = [(3+2\cos(t))\cos(s), (3+2\cos(t))\sin(s), s+2\sin(t)]$$

where $0 \leq s \leq 7\pi/2$ and $0 \leq t \leq 2\pi$.



Hint: This parametrization leads correctly already to a vector $\vec{r}_t \times \vec{r}_s$ pointing inwards. The boundary of the surface is made of two circles $\vec{r}(t,0)$ and $\vec{r}(t,7\pi/2)$. The picture gives the direction of the velocity vectors of these curves (which in each case might or might not be compatible with the orientation of the surface).

Solution:
We use Stokes theorem. Instead of computing the flux integral we compute the line integral along the two circles. The vector field was already split so that the second part is a gradient field. Both line integrals with that vector field are zero by the fundamental theorem of line integrals. The lower circle is already oriented correctly, the second one not. The first circle is obtained by putting $s=0$, the second one is obtained by putting $s = 7\pi/2$:

$$\vec{r}(t) = [3+2\cos(t)), 0, 2\sin(t)] \ ,$$

$$\vec{r}(t) = [0, -3-2\cos(t), 7\pi/2+2\sin(t)] \ .$$

The first line integral $\int_0^{2\pi}[-2\sin(t), 2\sin(t), 2\cos(t)] \cdot r'(t)dt = 8\pi$. The second line integral $\int_0^{2\pi}[7\pi/2+2\sin(t), 7\pi/2+2\sin(t), -3] \cdot r'[t]dt = 4\pi$ has to be taken negatively. The result is $8\pi - 4\pi = \boxed{4\pi}$.