Unit 21: Green's theorem

LECTURE

21.1. Green's theorem is the second integral theorem in two dimensions. In this unit, we do multivariable calculus in two dimensions, where we have only **two** derivatives, **two** integral theorems: the **fundamental theorem of line integrals** as well as **Green's theorem**. You might be used to think about the two-dimensional case as a special case of the xy-plane in three space, but we insist on remaining two dimensional. [1]

21.2. Remember that the **curl** of a vector field $\vec{F}(x,y) = [P(x,y), Q(x,y)]^T$ is the scalar field $\mathrm{curl}(F)(x,y) = \nabla \times \vec{F} = Q_x(x,y) - P_y(x,y)$. It measures the **vorticity** of the vector field at (x,y). For example, for $\vec{F}(x,y) = [x^3 + y^2, y^3 + x^2 y]^T$, we have $\mathrm{curl}(F)(x,y) = 2xy - 2y$.

> **Theorem:** **Green's theorem**: If $\vec{F}(x,y) = [P(x,y), Q(x,y)]^T$ is a vector field and G is a region for which the boundary C is a curve parametrized so that G is "to the left", then
> $$\int_C \vec{F} \cdot \vec{dr} = \int\int_G \mathrm{curl}(F) \, dxdy \ .$$

21.3. Take a square $G = [x, x+h] \times [y, y+h]$ with small $h > 0$. The line integral of $\vec{F} = [P, Q]^T$ along the boundary is $\int_0^h P(x+t, y)dt + \int_0^h Q(x+h, y+t) \, dt - \int_0^h P(x+t, y+h) \, dt - \int_0^h Q(x, y+t) \, dt$. It measures the "circulation" at the place (x,y). Because $Q(x+h, y) - Q(x,y) \sim Q_x(x,y)h$ and $P(x, y+h) - P(x,y) \sim P_y(x,y)h$, the line integral is $(Q_x - P_y)h^2$ is $\int_0^h \int_0^h \mathrm{curl}(F) \, dxdy$ with an error of the order h^3 or smaller. Now take a region G with area $|G|$ and chop it into small squares of size h. We need about $|G|/h^2$ such squares. Summing up all the line integrals around the boundaries is the sum of the line integral along the boundary of G because of the cancellations in the interior. On the boundary, it is a Riemann sum of the line integral along the boundary. The sum of the curls of the squares is a Riemann sum approximation of the double integral $\int\int_G \mathrm{curl}(F) \, dxdy$. Taking the limit $h \to 0$ gives Greens theorem.

---

[1] It is better to think about two dimensions as if we were flat-landers unaware about the third dimension. If we speak about "the plane", this is our universe, we are ignorant about 3 space. Edwin Abbot's Flatland is a 1884 romance plays in two dimensions.
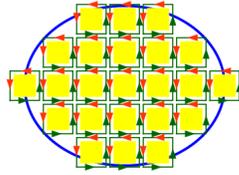
21.4. George Green lived from 1793 to 1841. Unfortunately, we don't have a single picture of him. He was a physicist, a self-taught mathematician as well as a miller. His work greatly contributed to modern physics.

21.5. Here is a special case: if $\vec{F}$ is a gradient field $\vec{F} = \nabla f$, then both sides of Green's theorem are zero: $\int_C \vec{F} \cdot d\vec{r}$ is zero by the fundamental theorem for line integrals. and $\int \int_G \text{curl}(F) \cdot dA$ is zero because $\text{curl}(F) = \text{curl}(\text{grad}(f)) = 0$.

21.6. If $\vec{F}(x, y) = \nabla f$ is a gradient field then the curl is zero because if $P(x, y) = f_x(x, y), Q(x, y) = f_y(x, y)$ and $\text{curl}(F) = Q_x - P_y = f_{yx} - f_{xy} = 0$ by Clairaut. $\vec{F}(x, y) = [x + y, yx]^T$ for example is not a gradient field because $\text{curl}(F) = y - 1$.

21.7. The already established **Clairaut identity**

$$\text{curl}(\text{grad}(f)) = 0$$

21.8. This can also be remembered by writing $\text{curl}(\vec{F}) = \nabla \times \vec{F}$ and $\text{curl}(\nabla f) = \nabla \times \nabla f$. Use now that cross product of two identical vectors is 0. Working with ∇ as a vector is called **nabla calculus** which can serve as a mnemonic.

21.9. It had been a consequence of the fundamental theorem of line integrals that:

If $\vec{F}$ is a gradient field then $\text{curl}(F) = 0$ everywhere.

21.10. Is the converse true? Here is the answer:

Definition: A region R is called **simply connected** if every closed loop in R can be pulled together continuously within R to a point inside R.

21.11. $R = \{x^2 + y^2 \leq 1\}$ is simply connected, $O = \{3 \leq x^2 + y^2 \leq 4\}$ is not.

If $\text{curl}(\vec{F}) = 0$ in a simply connected region G, then $\vec{F}$ is a gradient field.

Proof. Given a closed curve C in G enclosing a region R. Green's theorem assures that $\int_C \vec{F} \, d\vec{r} = 0$. So $\vec{F}$ has the closed loop property in G. This is equivalent to the fact that line integrals are path independent. In that case $\vec{F}$ is therefore a gradient field: one can get $f(x, y)$ by taking the line integral from an arbitrary point O to (x, y). In the homework, you look at an example of a not simply connected region where the $\text{curl}(\vec{F}) = 0$ does not imply that $\vec{F}$ is a gradient field.

<div align="center">EXAMPLES</div>

21.12. Problem: Find the line integral of $\vec{F}(x, y) = [x^2 - y^2, 2xy]^T = [P, Q]^T$ along the boundary of the rectangle $[0, 2] \times [0, 1]$. **Solution:** $\text{curl}(\vec{F}) = Q_x - P_y = 2y + 2y = 4y$ so that $\int_C \vec{F} \, d\vec{r} = \int_0^2 \int_0^1 4y \, dy dx = 2y^2|_0^1 x|_0^2 = 4$.

21.13. Problem: Find the area of the region enclosed by

$$\vec{r}(t) = [\frac{\sin(\pi t)^2}{t}, t^2 - 1]^T$$

for $-1 \le t \le 1$. To do so, use Greens theorem with the vector field $\vec{F} = [0, x]^T$.

21.14. Green's theorem allows to express the coordinates of the **centroid** = center of mass

$$(\int \int_G x \, dA/A, \int \int_G y \, dA/A)$$

using line integrals. With the vector field $\vec{F} = [0, x^2]^T$ we have

$$\int \int_G x \, dA = \int_C \vec{F} \, d\vec{r} \; .$$

21.15. An important application of Green is **area computation**: Take a vector field like $\vec{F}(x, y) = [P, Q]^T = [0, x]^T$ which has constant vorticity $\text{curl}(\vec{F})(x, y) = 1$. For $\vec{F}(x, y) = [0, x]^T$, the right hand side in Green's theorem is the **area** $\text{Area}(G) = \int_C \vec{F}(\vec{r}(t)) \, \vec{r}'(t) \, dt$.

21.16. Let G be the region below the graph of a function $f(x)$ on $[a, b]$. The line integral around the boundary of G is 0 from $(a, 0)$ to $(b, 0)$ because $\vec{F}(x, y) = [0, 0]^T$ there. The line integral is also zero from $(b, 0)$ to $(b, f(b))$ and $(a, f(a))$ to $(a, 0)$ because $N = 0$. The line integral along the curve $(t, f(t))$ is $-\int_a^b [-y(t), 0]^T \cdot [1, f'(t)]^T \, dt = \int_a^b f(t) \, dt$. Green's theorem confirms that this is the area of the region below the graph.

21.17. An engineering application is the **planimeter**, a mechanical device for measuring areas. We demonstrate it in class. Historically it had been used in medicine to measure the size of the cross-sections of tumors, in biology to measure the area of leaves or wing sizes of insects, in agriculture to measure the area of forests, in engineering to measure the size of profiles.

21.18. There is a vector field $\vec{F}$ associated to the device which is obtained by placing a unit vector perpendicular to the arm). One can prove that $\vec{F}$ has vorticity 1. The planimeter calculates the line integral of $\vec{F}$ along a given curve. Green's theorem assures this is the area.

Homework

This homework is due on Tuesday, 8/6/2019.

Problem 21.1: Given $f(x, y) = x^5 + xy^4$, compute the line integral of $\vec{F}(x, y) = [25y + 6y^2, 12xy + 10y^{44}]^T + \nabla f$ along the boundary of the **Monster region** given in the picture. There are four boundary curves, oriented as shown in the picture: a large ellipse of area 16, two circles of area 1 and 2 as well as a small ellipse (the mouth) of area 3.

Problem 21.2: Given $f(x, y) = x^5 + xy^4$, compute the line integral of $\vec{F}(x, y) = [15y + 6y^2, 12xy + y^4]^T + \nabla f$ along the boundary of the **Monster region** given in the picture. There are four boundary curves, oriented as shown in the picture: a large ellipse of area 16, two circles of area 1 and 2 as well as a small ellipse (the mouth) of area 3.

Problem 21.3: Find the area of the region bounded by the **hypocycloid** $\vec{r}(t) = [4\cos^3(t), 4\sin^3(t)]^T, 0 \le t \le 2\pi$.

Problem 21.4: Let G be the region $x^6 + y^6 \le 1$. Mathematica allows us to get the area as Area[ImplicitRegion[x^6 + y^6 <= 1, {x, y}]] and tells, it is $A = 3.8552$. Compute the line integral of $\vec{F}(x, y) = [x^{800} + \sin(x) - 55y, y^{12} + \cos(y) + 3x]^T$ along the boundary in terms of A (leave A in the answer).

Problem 21.5: Let C be the boundary curve of the white Yang part of the Ying-Yang symbol in the disc of radius 6. You can see in the image that the curve C has three parts, and that the orientation of each part is given. Find the line integral of the vector field

$$\vec{F}(x, y) = [-y + \sin(e^x), x]^T$$

around C.
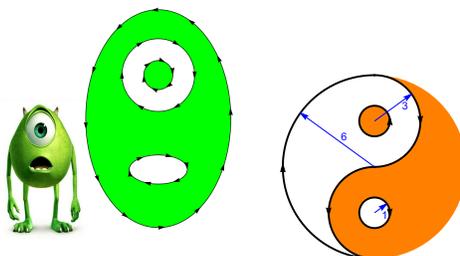
Unit 22: Curl and Flux

22.1. The **curl** in two dimensions was the scalar field $\text{curl}(F) = Q_x - P_y$. By Green's theorem, the curl evaluated at (x, y) is $\lim_{r \to 0} \int_{C_r} \vec{F} \, dr / (\pi r^2)$, where C_r is a small circle of radius r oriented counter clockwise an centered at (x, y). Green's theorem explains so what the curl is: it measures how the field "curls"... As rotations in two dimensions are determined by a single angle, in three dimensions, three parameters are needed. It is a vector whose direction tells the axes of rotation and the length tells the amount of rotation. The curl now becomes a vector:
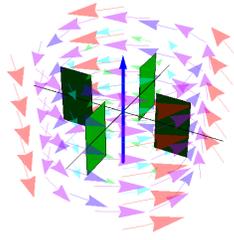
Definition: The **curl** of $\vec{F} = [P, Q, R]^T$ is the vector field
$$\text{curl}(P, Q, R) = [R_y - Q_z, P_z - R_x, Q_x - P_y]^T .$$

22.2. In **Nabla calculus**, this is written as $\text{curl}(\vec{F}) = \nabla \times \vec{F}$. Note that the third component $Q_x - P_y$ of the curl is for fixed z just the curl of the two-dimensional vector field $\vec{F} = [P, Q]^T$. While the curl in two dimensions is a scalar field, it is a vector field in 3 dimensions. In n dimensions, it would have $n(n-1)/2$ components, as this is the number of 2-dimensional coordinate planes. The curl measures the "vorticity" of the field and each component measures this in one of the two dimensional coordinate planes.

Definition: If a field has zero curl everywhere, the field is called **irrotational**.

22.3. The curl is frequently visualized using a "paddle wheel". If the rotation axes points into direction v, the rotation speed is $\vec{F} \cdot \vec{v}$. The direction in which the wheel turns fastest, is the direction of $\text{curl}(\vec{F})$. The angular velocity of the wheel is the length of the curl.

22.4. In two dimensions, we had two derivatives, the gradient and curl. In three dimensions, there are three fundamental derivatives: the **gradient**, the **curl** and the **divergence**.

> **Definition:** The **divergence** of $\vec{F} = [P, Q, R]^T$ is the scalar field $\mathrm{div}([P, Q, R]^T) = \nabla \cdot \vec{F} = P_x + Q_y + R_z$.

22.5. The divergence can also be defined in two dimensions, but it is there not as fundamental as it is not an "exterior derivatives". We want in d dimensions to have d fundamental derivatives and d fundamental integrals and d fundamental theorems. Distinguishing dimensions helps to organize the integral theorems. While Green looks like Stokes, we urge you to look at it as a different theorem taking place in "flatland". It is a small matter but it is much clearer to have in every dimension d a separate calculus. This prevents mixing up the theorems and makes things easier.

> **Definition:** In two dimensions, the **divergence** of $\vec{F} = [P, Q]^T$ is defined as $\mathrm{div}(P, Q) = \nabla \cdot \vec{F} = P_x + Q_y$.

22.6. In two dimensions, the divergence can be written as the curl of a -90 degrees rotated field $\vec{G} = [Q, -P]^T$ because $\mathrm{div}(\vec{G}) = Q_x - P_y = \mathrm{curl}(\vec{F})$. The divergence measures the "expansion" of a field. If a field has zero divergence everywhere, the field is called **incompressible**.

22.7. With the "vector" $\nabla = [\partial_x, \partial_y, \partial_z]^T$, we can write $\mathrm{curl}(\vec{F}) = \nabla \times \vec{F}$ and $\mathrm{div}(\vec{F}) = \nabla \cdot \vec{F}$. Formulating formulas using the "Nabla vector" and using rules from geometry is called **Nabla calculus**. This works both in 2 and 3 dimensions even so the ∇ vector is not an actual vector but an operator. The following combination of divergence and gradient often appears in physics:

> **Definition:**
> $$\Delta f = \mathrm{div}(\mathrm{grad}(f)) = f_{xx} + f_{yy} + f_{zz} .$$
> is called the **Laplacian** of f. One can write $\Delta f = \nabla^2 f$.

22.8. Mathematicians know Δ it as a 'form Laplacian". Here are some identities:

$\operatorname{div}(\operatorname{curl}(\vec{F})) = 0.$
$\operatorname{curlgrad}(\vec{F}) = \vec{0}$
$\operatorname{curl}(\operatorname{curl}(\vec{F})) = \operatorname{grad}(\operatorname{div}(\vec{F}) - \Delta(\vec{F})).$

EXAMPLES

22.9. Question: Is there a vector field $\vec{G}$ such that $\vec{F} = [x + y, z, y^2]^T = \operatorname{curl}(\vec{G})$?
Answer: No, because $\operatorname{div}(\vec{F}) = 1$ is incompatible with $\operatorname{div}(\operatorname{curl}(\vec{G})) = 0$.

22.10. Show that in simply connected region, every irrotational and incompressible field can be written as a vector field $\vec{F} = \operatorname{grad}(f)$ with $\Delta f = 0$. Proof. Since $\vec{F}$ is irrotational, there exists a function f satisfying $F = \operatorname{grad}(f)$. As the region is simply connected, we can deform any path between two points without changing the result. Now, $\operatorname{div}(F) = 0$ implies $\operatorname{divgrad}(f) = \Delta f = 0$.

22.11. If we rotate the vector field $\vec{F} = [P, Q]^T$ by 90 degrees $= \pi/2$, we get a new vector field $\vec{G} = [-Q, P]^T$. The integral $\int_C F \cdot ds$ becomes a **flux** $\int_\gamma G \cdot dn$ of G through the boundary of R, where dn is a normal vector with length $|r'|dt$. With $\operatorname{div}(\vec{F}) = (P_x + Q_y)$, we see that

$$\operatorname{curl}(\vec{F}) = \operatorname{div}(\vec{G}) \ .$$

Green's theorem now becomes

$$\int\int_R \operatorname{div}(\vec{G}) \, dxdy = \int_C \vec{G} \cdot \vec{dn} \ ,$$

where $dn(x, y)$ is a normal vector at (x, y) orthogonal to the velocity vector $\vec{r}\,'(x, y)$ at (x, y). This new theorem has a generalization to three dimensions, where it is called Gauss theorem or divergence theorem. Don't treat this however as a different theorem in two dimensions. It is just Green's theorem in disguise.

In two dimensions, the divergence at a point (x, y) is the average flux of the field through a small circle of radius r around the point in the limit when the radius of the circle goes to zero.

We have now all the derivatives we need. In dimension d, there are d fundamental derivatives.
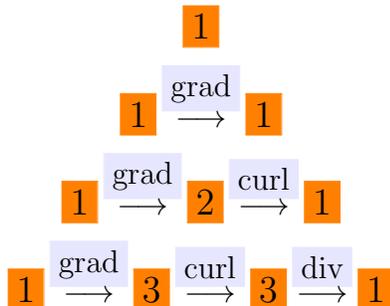
Homework

This homework is due on Tuesday, 8/6/2019.

Problem 22.1:   Construct your own nonzero vector field $\vec{F}(x,y) = [P(x,y), Q(x,y)]^T$ in each of the following cases:
a) $\vec{F}$ is irrotational but not incompressible.
b) $\vec{F}$ is incompressible but not irrotational.
c) $\vec{F}$ is irrotational and incompressible.
d) $\vec{F}$ is not irrotational and not incompressible.

Problem 22.2:   The vector field $\vec{F}(x,y,z) = [x,y,-2z]^T$ satisfies $\operatorname{div}(\vec{F}) = 0$. Can you find a vector field $\vec{G}(x,y,z)$ such that $\operatorname{curl}(\vec{G}) = \vec{F}$? Such a field $\vec{G}$ is called a **vector potential**.
Hint. Write $\vec{F}$ as a sum $[x,0,-z]^T + [0,y,-z]^T$ and find vector potentials for each of the parts using a vector field you have seen on the blackboard in class.

Problem 22.3:   Evaluate the flux integral $\int\int_S [0,0,yz]^T \cdot d\vec{S}$, where S is the surface with parametric equation $x = uv, y = u+v, z = u - v$ on $R : u^2 + v^2 \leq 4$ and $u > 0$.

Problem 22.4:   Evaluate the flux integral $\int\int_S \operatorname{curl}(F) \cdot d\vec{S}$ for
$$\vec{F}(x,y,z) = [3xy, 3yz, 3zx]^T .$$
where S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $[0,2] \times [0,2]$ and has an upward orientation.

Problem 22.5:   a) What is the relation between the flux of the vector field $\vec{F} = \nabla g/|\nabla g|$ through the surface $S : \{g = 1\}$ with $g(x,y,z) = x^6 + y^4 + 2z^8$ and the surface area of S?
b) Find the flux of the vector field $\vec{G} = \nabla g \times [0,0,2]^T$ through the surface S.

1

---

[1]Both a and b) do not need any computation. You can answer each question with one sentence. In part a) compare $\vec{F} \cdot d\vec{S}$ with dS in that case.

Unit 23: Stokes Theorem

22.1. We work with a surface S parametrized as $\vec{r}(u,v) = [x(u,v), y(u,v), z(u,v)]^T$ over a domain R in the uv-plane. Remember that the **flux integral** of $\vec{F}$ through S is defined as the double integral

$$\iint_R \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) \, dudv .$$

The following theorem is the second fundamental theorem of calculus in three dimensions:

> **Definition:** The **boundary** of a surface S consists of all points P where even arbitrary small circle $S_r(P) \cap S$ around the point is not closed.

22.2. The boundary is a collection of curves oriented so that the surface is to the "left" if the normal vector to the surface is pointing "up". In other words, the velocity vector v, a vector w pointing towards the surface and the normal vector n to the surface form a right handed coordinate system.

> **Theorem:** **Stokes theorem**: Let S be a surface bounded by a curve C and $\vec{F}$ be a vector field. Then
> $$\iint_S \operatorname{curl}(\vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} .$$
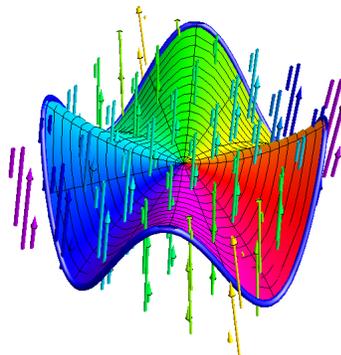
Proof. Stokes theorem is proven in the same way than Green's theorem. Chop up S into a union of small triangles. As before, the sum of the fluxes through all these triangles adds up to the flux through the surface and the sum of the line integrals along the boundaries adds up to the line integral of the boundary of S. Stokes theorem for a small triangle can be reduced to Green's theorem because with a coordinate system such that the triangle is in the xy plane, the flux of the field is the double integral of $\text{curl}\vec{F}\ d\vec{S} = \text{curl}F(\vec{r}) \cdot \vec{n}dudv = (Q_x - P_y)\cos(\theta)dudv$, where θ is the angle between the normal vector and $\vec{F} = [P, Q, R]^T$. On the other hand, since the power $\vec{F}(\vec{r}) \cdot \vec{r}'(t)dt = (P(\vec{r})\cos(\theta)x'(t) + Q(\vec{r})\cos(\theta)y'(t))dt$ also has everything multiplied by $\cos(\theta)$, the result for each space triangle follows from Green. Stokes theorem now follows by making the triangulation finer and finer. On both sides we have a Riemann sum approximation to the integrals. $\square$

Examples

22.3. Let $\vec{F}(x, y, z) = [-y, x, 0]^T$ and let S be the upper semi hemisphere, then $\text{curl}(\vec{F})(x, y, z) = [0, 0, 2]^T$. The surface is parameterized by

$$\vec{r}(u, v) = [\cos(u)\sin(v), \sin(u)\sin(v), \cos(v)]^T$$

on $R = [0, 2\pi] \times [0, \pi/2]$ and $\vec{r}_u \times \vec{r}_v = \sin(v)\vec{r}(u, v)$ so that $\text{curl}(\vec{F})(x, y, z) \cdot \vec{r}_u \times \vec{r}_v = \cos(v)\sin(v)2$. The integral $\int_0^{2\pi} \int_0^{\pi/2} \sin(2v)\ dvdu = 2\pi$.

The boundary C of S is parameterized by $\vec{r}(t) = [\cos(t), \sin(t), 0]^T$ so that $d\vec{r} = \vec{r}\,'(t)dt = [-\sin(t), \cos(t), 0]^T\ dt$ and $\vec{F}(\vec{r}(t))\ \vec{r}\,'(t)dt = \sin(t)^2 + \cos^2(t) = 1$. The line integral $\int_C \vec{F} \cdot d\vec{r}$ along the boundary is 2π.

22.4. If S is a surface in the xy-plane and $\vec{F} = [P, Q, 0]^T$ has zero z component, then $\text{curl}(\vec{F}) = [0, 0, Q_x - P_y]^T$ and $\text{curl}(\vec{F}) \cdot d\vec{S} = Q_x - P_y\ dxdy$. We see that for a surface which is flat, Stokes theorem is a consequence of Green's theorem. If we put the coordinate axis so that the surface is in the xy-plane, then the vector field F induces a vector field on the surface such that its $2D$ curl is the normal component of $\text{curl}(F)$. The reason is that the third component $Q_x - P_y$ of $\text{curl}(\vec{F})[R_y - Q_z, P_z - R_x, Q_x - P_y]^T$ is the two dimensional curl: $\vec{F}(\vec{r}(u, v)) \cdot [0, 0, 1]^T = Q_x - P_y$. If C is the boundary of the surface, then $\int \int_S \vec{F}(\vec{r}(u, v)) \cdot [0, 0, 1]^T\ dudv = \int_C \vec{F}(\vec{r}(t))\ \vec{r}\,'(t)dt$.

22.5. Calculate the flux of the curl of $\vec{F}(x, y, z) = [-y, x, 0]^T$ through the surface parameterized by $\vec{r}(u, v) = [\cos(u)\cos(v), \sin(u)\cos(v), \cos^2(v) + \cos(v)\sin^2(u + \pi/2)]^T$. Because the surface has the same boundary as the upper half sphere, the integral is again 2π as in the above example.

22.6. For every surface bounded by a curve C, the flux of $\text{curl}(\vec{F})$ through the surface is the same. Proof. The flux of the curl of a vector field through a surface S depends only on the boundary of S. Compare this with the earlier statement that for every curve between two points A, B the line integral of $\text{grad}(f)$ along C is the same. The line integral of the gradient of a function of a curve C depends only on the end points of C.

22.7. Electric and magnetic fields are linked by the **Maxwell equation** $\text{curl}(\vec{E}) = -\frac{1}{c}\dot{B}$. These are examples of partial differential equations. If a closed wire C bounds a surface S then $\iint_S B \cdot dS$ is the flux of the magnetic field through S. Its change can be related with a voltage using Stokes theorem: $d/dt \iint_S B \cdot dS = \iint_S \dot{B} \cdot dS = \iint_S -c \text{ curl}(\vec{E}) \cdot d\vec{S} = -c \int_C \vec{E} \; d\vec{r} = U$, where U is the voltage. If we change the flux of the magnetic field through the wire, then this induces a voltage. The flux can be changed by changing the amount of the magnetic field but also by changing the direction. If we turn around a magnet around the wire or the wire inside the magnet, we get an electric voltage. This happens in a power-generator, like the alternator in a car. Stokes theorem explains why we can generate electricity from motion.
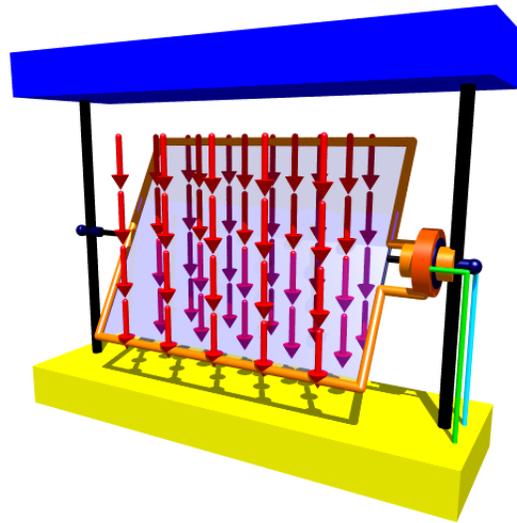
22.8. The history of Stokes theorem is a bit hazy. [1]. A version of Stokes theorem appeared to be known by **André Ampère** in 1825. **William Thomson** (Lord Kelvin) mentioned the theorem to Stokes in 1850). **George Gabriel Stokes** (1819-1903) who found parts of the identity earlier 1840 formulated it in a prize exam from 1854 (the proof is one of the exam problems). The first pushed proof is by Hermann Hankel in 1861.
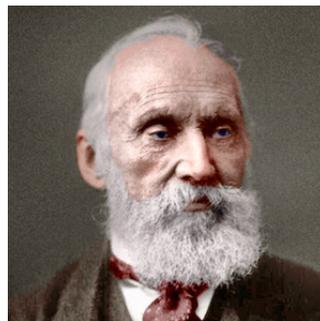
George Gabriel Stokes   William Thomson (Kelvin)   André Marie Ampere

---

[1]See V. Katz, the History of Stokes theorem, Mathematics Magazine 52, 1979, p 146-156

HOMEWORK

This homework is due on Tuesday, 8/6/2019.

Problem 23.1: Assume S is the surface $x^8 + y^4 + z^6 = 100$ and $\vec{F} = [e^{e^{xyz}}, x^2yz, x - y - \sin(zx)]^T$. Explain why $\iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = 0$.

Problem 23.2: Find $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x, y, z) = [12x^2y, 4x^3, 12xy]^T$ and C is the curve of intersection of the hyperbolic paraboloid $z = y^2 - x^2$ and the cylinder $x^2 + y^2 = 1$, oriented counterclockwise as viewed from above.

Problem 23.3: Evaluate the flux integral $\int \int_S \text{curl}(\vec{F}) \cdot d\vec{S}$, where

$$\vec{F}(x, y, z) = [xe^{y^2}z^3 + 2xyze^{x^2+z}, x + z^2e^{x^2+z}, ye^{x^2+z} + ze^x]^T$$

and where S is the part of the ellipsoid $x^2 + y^2/4 + (z + 1)^2 = 2$, $z > 0$ oriented so that the normal vector points upwards.

Problem 23.4: Find the line integral $\int_C \vec{F} \, d\vec{r}$, where C is the circle of radius 5 in the xz-plane oriented counter clockwise when looking from the point $(0, 1, 0)$ onto the plane and where $\vec{F}$ is the vector field

$$\vec{F}(x, y, z) = [9x^2z + x^5, \cos(e^y), -9xz^2 + \sin(\sin(z)]^T .$$

Use a convenient surface S which has C as a boundary.

Problem 23.5: Find the flux integral $\iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$, where $\vec{F}(x, y, z) =$

$$[2\cos(\pi y)e^{2x} + z^2, x^2\cos(z\pi/2) - \pi \sin(\pi y)e^{2x}, 2xz]^T$$

and S is the surface parametrized by

$$\vec{r}(s, t) = [(1 - s^{1/3})\cos(t) - 4s^2, (1 - s^{1/3})\sin(t), 5s]^T$$

with $0 \le t \le 2\pi, 0 \le s \le 1$ and oriented so that the normal vectors point to the outside of the thorn.
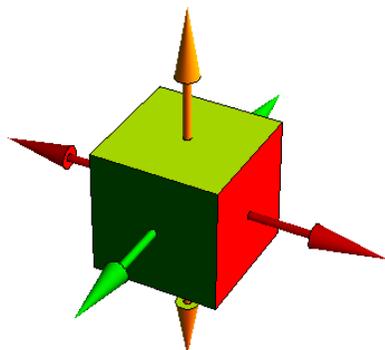
Unit 24: Divergence Theorem

22.1. There are three integral theorems in three dimensions. We have already seen the fundamental theorem of line integrals and Stokes theorem. The **divergence theorem** completes the list of integral theorems in three dimensions:

> **Theorem: Divergence Theorem**. If E be a solid with boundary surface S oriented so that the normal vector points outside and if $\vec{F}$ be a vector field, then
> $$\iiint_E \operatorname{div}(\vec{F}) \, dV = \iint_S \vec{F} \cdot dS \ .$$

22.2. To prove this, take a small box $[x, x+dx] \times [y, y+dy] \times [z, z+dz]$. The flux of $\vec{F} = [P, Q, R]^T$ through the faces perpendicular to the x-axes is $[\vec{F}(x+dx, y, z) \cdot [1, 0, 0]^T + \vec{F}(x, y, z) \cdot [-1, 0, 0]^T]dydz = P(x+dx, y, z) - P(x, y, z) \sim P_x \, dxdydz$. Similarly, the flux through the y-boundaries is $P_y \, dydxdz$ and the flux through the two z-boundaries is $P_z \, dzdxdy$. The total flux through the faces of the cube is $(P_x + P_y + P_z) \, dxdydz = \operatorname{div}(\vec{F}) \, dxdydz$. A general solid can be approximated as a union of small cubes. The sum of the fluxes through all the cubes consists now of the flux through all faces without neighboring faces. Important is that fluxes through adjacent sides cancel. The sum of all the fluxes of the cubes is the flux through the boundary of the union. The sum of all the $\operatorname{div}(\vec{F}) \, dxdydz$ is a Riemann sum approximation for the integral $\iiint_G \operatorname{div}(\vec{F}) \, dxdydz$. In the limit, when dx, dy, dz all go to zero, we obtain the divergence theorem.

22.3. The theorem explains what divergence means. If we integrate the divergence over a small cube, it is equal the flux of the field through the boundary of the cube. If this is positive, then more field exits the cube than entering the cube. There is field "generated" inside. The divergence measures the "expansion" of the field.

Examples

22.4. Let $\vec{F}(x, y, z) = [x, y, z]^T$ and let S be the unit sphere. The divergence of $\vec{F}$ is the constant function $\operatorname{div}(\vec{F}) = 3$ and $\iiint_G \operatorname{div}(\vec{F})\, dV = 3 \cdot 4\pi/3 = 4\pi$. The flux through the boundary is $\iint_S \vec{r} \cdot \vec{r}_u \times \vec{r}_v\, dudv = \iint_S |\vec{r}(u, v)|^2 \sin(v)\, dudv = \int_0^\pi \int_0^{2\pi} \sin(v)\, dudv = 4\pi$ also. We see that the divergence theorem allows us to compute the area of the sphere from the volume of the enclosed ball or compute the volume from the surface area.

22.5. What is the flux of the vector field $\vec{F}(x, y, z) = [2x, 3z^2 + y, \sin(x)]^T$ through the solid $G = [0, 3] \times [0, 3] \times [0, 3] \setminus ([0, 3] \times [1, 2] \times [1, 2] \cup [1, 2] \times [0, 3] \times [1, 2] \cup [0, 3] \times [0, 3] \times [1, 2])$ which is a cube where three perpendicular cubic holes have been removed? **Solution:** Use the divergence theorem: $\operatorname{div}(\vec{F}) = 2$ and so $\iiint_G \operatorname{div}(\vec{F})\, dV = 2 \iiint_G\, dV = 2\operatorname{Vol}(G) = 2(27 - 7) = 40$. Note that the flux integral here would be over a complicated surface over dozens of rectangular planar regions.

22.6. Find the flux of $\operatorname{curl}(F)$ through a torus if $\vec{F} = [yz^2, z + \sin(x) + y, \cos(x)]^T$ and the torus has the parametrization

$$\vec{r}(\theta, \phi) = [(2 + \cos(\phi))\cos(\theta), (2 + \cos(\phi))\sin(\theta), \sin(\phi)]^T .$$

Solution: The answer is 0 because the divergence of $\operatorname{curl}(F)$ is zero. By the divergence theorem, the flux is zero.

22.7. Similarly as Green's theorem allowed to calculate the area of a region by passing along the boundary, the volume of a region can be computed as a flux integral: Take for example the vector field $\vec{F}(x, y, z) = [x, 0, 0]^T$ which has divergence 1. The flux of this vector field through the boundary of a solid region is equal to the volume of the solid: $\iint_{\delta G}[x, 0, 0]^T \cdot d\vec{S} = \text{Vol}(G)$.
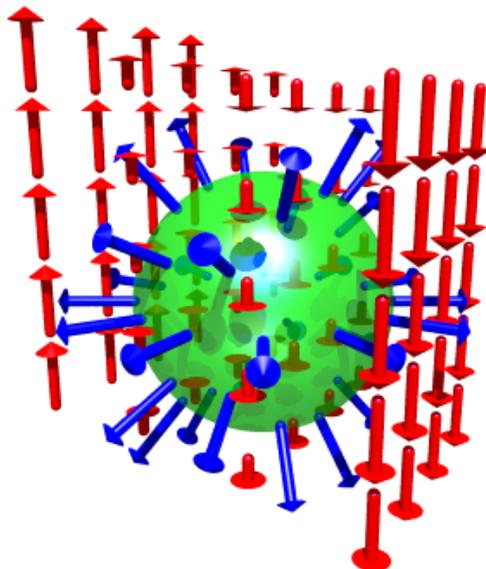
22.8. How heavy are we, at distance r from the center of the earth?
Solution: The law of gravity can be formulated as $\text{div}(\vec{F}) = 4\pi\rho$, where ρ is the mass density. We assume that the earth is a ball of radius R. By rotational symmetry, the gravitational force is normal to the surface: $\vec{F}(\vec{x}) = \vec{F}(r)\vec{x}/||\vec{x}||$. The flux of $\vec{F}$ through a ball of radius r is $\iint_{S_r} \vec{F}(x) \cdot d\vec{S} = 4\pi r^2 \vec{F}(r)$. By the **divergence theorem**, this is $4\pi M_r = 4\pi \iiint_{B_r} \rho(x) \ dV$, where M_r is the mass of the material inside S_r. We have $(4\pi)^2 \rho r^3/3 = 4\pi r^2 \vec{F}(r)$ for $r < R$ and $(4\pi)^2 \rho R^3/3 = 4\pi r^2 \vec{F}(r)$ for $r \geq R$. Inside the earth, the gravitational force $\vec{F}(r) = 4\pi \rho r/3$. Outside the earth, it satisfies $\vec{F}(r) = M/r^2$ with $M = 4\pi R^3 \rho/3$.
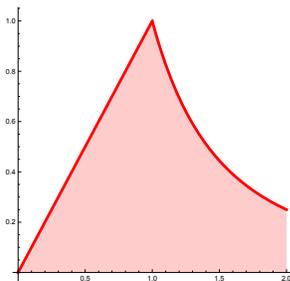
22.9. To the end we make an overview over the integral theorems and give an other typical example in each case.

The fundamental theorem for line integrals, Green's theorem, Stokes theorem and divergence theorem are all part of **one single theorem** $\int_A dF = \int_{\delta A} F$, where dF is a **exterior derivative** of F and where δA is the **boundary** of A. It generalizes the fundamental theorem of calculus.

> **Fundamental theorem of line integrals**: If C is a curve with boundary $\{A, B\}$ and f is a function, then
> $$\int_C \nabla f \cdot \vec{dr} = f(B) - f(A)$$

Remarks.

1) For closed curves, the line integral $\int_C \nabla f \cdot \vec{dr}$ is zero.
2) Gradient fields are **path independent**: if $\vec{F} = \nabla f$, then the line integral between two points P and Q does not depend on the path connecting the two points.
3) The theorem holds in any dimension. In one dimension, it reduces to the **fundamental theorem of calculus** $\int_a^b f'(x)\, dx = f(b) - f(a)$
4) The theorem justifies the name **conservative** for gradient vector fields.
5) The term "potential" was coined by George Green who lived from 1783-1841.

22.10. Example. Let $f(x, y, z) = x^2 + y^4 + z$. Find the line integral of the vector field $\vec{F}(x, y, z) = \nabla f(x, y, z)$ along the path $\vec{r}(t) = [\cos(5t), \sin(2t), t^2]^T$ from $t = 0$ to $t = 2\pi$.

Solution. $\vec{r}(0) = [1, 0, 0]^T$ and $\vec{r}(2\pi) = [1, 0, 4\pi^2]^T$ and $f(\vec{r}(0)) = 1$ and $f(\vec{r}(2\pi)) = 1 + 4\pi^2$. The fundamental theorem of line integral gives $\int_C \nabla f\, \vec{dr} = f(r(2\pi)) - f(r(0)) = 4\pi^2$.

> **Green's theorem**. If R is a region with boundary C and $\vec{F}$ is a vector field, then
> $$\iint_R \operatorname{curl}(\vec{F})\, dxdy = \int_C \vec{F} \cdot \vec{dr} \ .$$

22.11. Remarks.
1) Greens theorem allows to switch from double integrals to one dimensional integrals.
2) The curve is oriented in such a way that the region is to the left.
3) The boundary of the curve can consist of piecewise smooth pieces.
4) If $C : t \mapsto \vec{r}(t) = [x(t), y(t)]^T$, the line integral is $\int_a^b [P(x(t), y(t)), Q(x(t), y(t))]^T \cdot [x'(t), y'(t)]^T\, dt$.
5) Green's theorem was found by George Green (1793-1841) in 1827 and by Mikhail Ostrogradski (1801-1862).
6) If $\operatorname{curl}(\vec{F}) = 0$ in a simply connected region, then the line integral along a closed curve is zero. If two curves connect two points then the line integral along those curves agrees.

7) Taking $\vec{F}(x,y) = [-y, 0]^T$ or $\vec{F}(x,y) = [0, x]^T$ gives **area formulas**.

22.12. Example. Find the line integral of the vector field $\vec{F}(x,y) = [x^4 + \sin(x) + y, x + y^3]^T$ along the path $\vec{r}(t) = [\cos(t), 5\sin(t) + \log(1 + \sin(t))]^T$, where t runs from $t = 0$ to $t = \pi$.

Solution. $\mathrm{curl}(\vec{F}) = 0$ implies that the line integral depends only on the end points $(0,1), (0,-1)$ of the path. Take the simpler path $\vec{r}(t) = [-t, 0]^T, -1 \leq t \leq 1$, which has velocity $\vec{r}\,'(t) = [-1, 0]^T$. The line integral is $\int_{-1}^{1}[t^4 - \sin(t), -t]^T \cdot [-1, 0]^T \ dt = -t^5/5|_{-1}^{1} = -2/5$.

Remark We could also find a potential $f(x,y) = x^5/5 - \cos(x) + xy + y^5/4$. It has the property that $\mathrm{grad}(f) = F$. Again, we get $f(0,-1) - f(0,1) = -1/5 - 1/5 = -2/5$.

> **Stokes theorem**. If S is a surface with boundary C and $\vec{F}$ is a vector field, then
>
> $$\iint_S \mathrm{curl}(\vec{F}) \cdot dS = \int_C \vec{F} \cdot d\vec{r} \ .$$

22.13. Remarks.
1) Stokes theorem allows to derive Greens theorem: if $\vec{F}$ is z-independent and the surface S is contained in the xy-plane, one obtains the result of Green.
2) The orientation of C is such that if you walk along C and have your head in the direction of the normal vector $\vec{r}_u \times \vec{r}_v$, then the surface to your left.
3) Stokes theorem was found by André Ampère (1775-1836) in 1825 and rediscovered by George Stokes (1819-1903).
4) The flux of the curl of a vector field does not depend on the surface S, only on the boundary of S.
5) The flux of the curl through a closed surface like the sphere is zero: the boundary of such a surface is empty.

22.14. Example. Compute the line integral of $\vec{F}(x,y,z) = [x^3 + xy, y, z]^T$ along the polygonal path C connecting the points $(0,0,0), (2,0,0), (2,1,0), (0,1,0)$ in that order.
Solution. The path C bounds a surface $S : \vec{r}(u,v) = [u,v,0]^T$ parameterized by $R = [0,2] \times [0,1]$. By Stokes theorem, the line integral is equal to the flux of $\mathrm{curl}(\vec{F})(x,y,z) = [0,0,-x]^T$ through S. The normal vector of S is $\vec{r}_u \times \vec{r}_v = [1,0,0]^T \times [0,1,0]^T = [0,0,1]^T$ so that $\iint_S \mathrm{curl}(\vec{F}) \ d\vec{S} = \int_0^2 \int_0^1 [0,0,-u]^T \cdot [0,0,1]^T \ dudv = \int_0^2 \int_0^1 -u \ dudv = -2$.

> **Divergence theorem**: If S is the boundary of a region E in space and $\vec{F}$ is a vector field, then
> $$\iiint_B \operatorname{div}(\vec{F})\, dV = \iint_S \vec{F} \cdot d\vec{S} \ .$$

**22.15. Remarks**.
1) The divergence theorem is also called **Gauss theorem**.
2) It is useful to determine the flux of vector fields through surfaces.
3) It can be used to compute volume.
3) It was discovered in 1764 by Joseph Louis Lagrange (1736-1813), later it was redis-covered by Carl Friedrich Gauss (1777-1855) and by George Green.
4) For divergence free vector fields $\vec{F}$, the flux through a closed surface is zero. Such fields $\vec{F}$ are also called **incompressible** or **source free**.

22.16. Example. Compute the flux of the vector field $\vec{F}(x,y,z) = [-x, y, z^2]^T$ through the boundary S of the rectangular box $[0,3] \times [-1,2] \times [1,2]$.
Solution. By Gauss theorem, the flux is equal to the triple integral of $\operatorname{div}(F) = 2z$ over the box: $\int_0^3 \int_{-1}^2 \int_1^2 2z\, dxdydz = (3-0)(2-(-1))(4-1) = 27$.

22.17. How do these theorems fit together? In n-dimensions, there are n theorems. We have here seen the situation in dimension $n = 2$ and $n = 3$, but one could continue. The fundamental theorem of line integrals generalizes directly to higher dimensions. Also the divergence theorem generalizes directly since an n-dimensional integral in n dimensions. The generalization of curl and flux would need more explanation as for $n = 4$ already, the curl of a vector field is a 6-dimensional object. It is a $n(n-1)/2$ dimensional object in general.
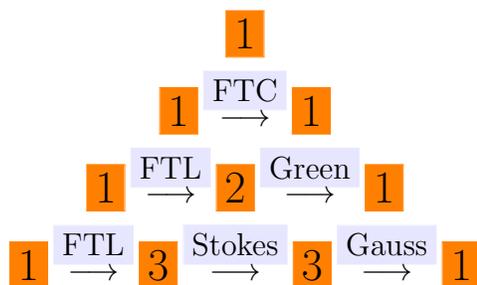
22.18. In one dimension, there is one derivative $f(x) \to f'(x)$ from scalar to scalar functions. It corresponds to the entry $1 - 1$ in the Pascal triangle. The next entry $1 - 2 - 1$ corresponds to differentiation in two dimensions, where we have the gradient $f \to \nabla f$ mapping a scalar function to a vector field with 2 components as well as the curl, $F \to \operatorname{curl}(F)$ which corresponds to the transition $2 - 1$. The situation in three dimensions is captured by the entry $1 - 3 - 3 - 1$ in the Pascal triangle. The first derivative $1 - 3$ is the gradient. The second derivative $3 - 3$ is the curl and the third derivative $3 - 1$ is the divergence. In $n = 4$ dimensions, we would have to look at $1 - 4 - 6 - 4 - 1$. The first derivative $1 - 4$ is still the gradient. Then we have a first

curl, which maps a vector field with 4 components into an object with 6 components. Then there is a second curl, which maps an object with 6 components back to a vector field, we would have to look at $1 - 4 - 6 - 4 - 1$.

22.19. When setting up calculus in dimension n, one talks about **differential forms** instead of scalar fields or vector fields. Functions are 0 forms or n-forms. Vector fields can be described by 1 or $n - 1$ forms. The general formalism defines a derivative d called **exterior derivative** on differential forms. It maps k forms to $k+1$ forms. There is also an integration of k-forms on k-dimensional objects. The **boundary operation** δ which maps a k-dimensional object into a $k - 1$ dimensional object. This boundary operation is dual to differentiation. They both satisfy the same relation $dd(F) = 0$ and $\delta\delta G = 0$. Differentiation and integration are linked by the general Stokes theorem:

$$\int_{\delta G} F = \int_G dF$$

22.20. One can see this as a single theorem, the **fundamental theorem of multivariable calculus**. The theorem is simpler in quantum calculus, where geometric objects and fields are on the same footing. There are various ways how one can generalize this. One way is to write it as $< \delta G, F > = < G, dF >$ which in linear algebra would be written as $[A^T v, w]^T = [v, Aw]^T$, where A^T is the transpose of a matrix A $[v, w]^T$ is the dot product. Since traditional calculus we deal with "smooth" functions and fields, we have to pay a prize and consider in turn "singular" objects like points or curves and surfaces. These are idealized objects which have zero diameter, radius or thickness.

22.21. So, it is all about **geometries** and **fields**. Geometries are curves, or surfaces or solids. Fields are scalar functions or vector fields. Geometries G can be "differentiated" by taking the boundary δG. Fields F can be differentiated by applying differential operators dF like grad, curl or div. And then there is integration which pairs up geometries G and fields F. The fundamental theorem $\int_{\delta G} F = \int_G dF$ tells that taking the boundary on the object corresponds to taking the derivative of the field.

22.22. Nature likes simplicity and elegance [1] and therefore found a quantum mathematics to be more fundamental. But the symmetry in which **geometry and fields become indistinguishable** manifests only in the very small.

---

[1]Leibniz: 1646-1716

Homework

This homework is due on Tuesday, 8/6/2019.

Problem 24.1: Compute using the divergence theorem the flux of the vector field $\vec{F}(x, y, z) = [9y, 2xy, 4yz + 187xy]^T$ through the unit cube $[0, 1] \times [0, 1] \times [0, 1]$.

Problem 24.2: Find the flux of the vector field $\vec{F}(x, y, z) = [xy, yz, zx]^T$ through the cylinder $x^2 + y^2 \leq 1$, $0 < z \leq 2$ without the bottom disk $z = 0$. (Still use the divergence theorem by closing it and computing the flux through the bottom.)

Problem 24.3:   Use the divergence theorem to calculate the flux of $\vec{F}(x, y, z) = [x^3, y^3, z^3]^T$ through the sphere $S : x^2 + y^2 + z^2 = 1$, where the sphere is oriented so that the normal vector points outwards.

Problem 24.4: Assume the vector field
$$\vec{F}(x, y, z) = [5x^3 + 12xy^2, y^3 + e^y \sin(z), 5z^3 + e^y \cos(z)]^T$$
is the magnetic field of the **sun** whose surface is a sphere of radius 3 oriented with the outward orientation. Compute the magnetic flux $\iint_S \vec{F} \cdot d\vec{S}$.

Problem 24.5: Find $\int \int_S \vec{F} \cdot d\vec{S}$, where $\vec{F}(x, y, z) = [-37x, 22y, 25z]^T$ and S is the boundary of the solid built with the 18 cubes shown in the picture.