MULTIVARIABLE CALCULUS

MATH S-21A

Unit 17: Triple integrals

Lecture

17.1. Integrating over higher dimensional regions is done in the same way than in two dimensions. Three dimensional regions are referred to as **solids**.

Definition: If $f(x, y, z)$ is continuous and E is a **bounded solid** in $\mathbb{R}^3$, then $\iint_E f(x, y, z)\ dxdydz$ is defined as the $n \to \infty$ limit of the Riemann sum

$$\frac{1}{n^3} \sum_{(\frac{i}{n}, \frac{j}{n}, \frac{k}{n}) \in E} f(\frac{i}{n}, \frac{j}{n}, \frac{k}{n})\ .$$

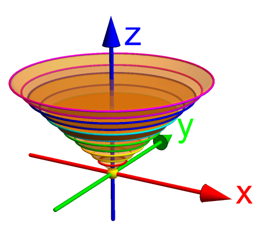
Triple integrals can be evaluated by iterated single integrals:

17.2. If E is the box $\{x \in [1, 2], y \in [0, 1], z \in [0, 1]\}$ and $f(x, y, z) = 24x^2y^3z$.

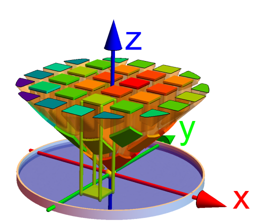
$$\int_0^1 \int_0^1 \int_0^1 24x^2y^3z\ dz\ dy\ dx.$$

To evaluate the integral, start from the inside $\int_0^1 24x^2y^3z\ dz = 12x^3y^3$, then then integrate the middle layer, $\int_0^1 12x^3y^3\ dy = 3x^2$ and finally and finally handle the most outer layer: $\int_1^2 3x^2 dx = 7$.
For the inner integral, $x = x_0$ and $y = y_0$ are fixed. The middle integral now computes the contribution over a slice $z = z_0$ intersected with R. The outer integral sums up all these slice contributions.

17.3. There are two reductions possible to compute triple integrals:



The **burger method** slices the solid a line and computes $\int_a^b \iint_{R(z)} f(x, y, z)\ dA\ dz$, where $g(z)$ is a double integral giving the values when integrating over cheese, meet or tomato. The **fries method** eats up fries going from $g(x, y)$ to $h(x, y)$ over a region R. We have $\iint_R [\int_{g(x,y)}^{h(x,y)} f(x, y, z)\ dz]\ dA$.



17.4. A special case is the **signed volume**

$$\int\int_R \int_0^{f(x,y)} 1 \; dzdxdy \; .$$

below the graph of a function $f(x,y)$ and above a region R, considered part of the xy-plane. It is the integral $\int\int_R f(x,y) \; dA$. The triple integral which is more natural when considering physical units as volume is measured in cubic meters for example. The triple integral also allows for flexibility: we can replace 1 with a function $f(x,y,z)$. If interpreted as a **charge density**, then the integral is the total charge.

17.5. The problem of computing volumes has been tackled early. **Archimedes (287-212 BC)** already developed an integration method which allowed him to find areas, volumes and surface areas in many cases without calculus. His method of **exhaustion** is close to the numerical method of integration by Riemann sum. In our terminology, Archimedes used the **washer method** to reduce the problem to a single variable problem. The **Archimedes principle** states that any body submerged in a water is acted upon by an upward force which is equal to the weight of the displaced water. This provides a practical way to compute volumes of complicated bodies. A second method, the **displacement method** is a **comparison technique**: the area of a sphere is the area of the cylinder enclosing it. The volume of a sphere is the volume of the complement of a cone in that cylinder. Modern rearrangement techniques use this still today in modern analysis. Heureka! **Cavalieri (1598-1647)** would build on Archimedes ideas and determine area and volume using tricks now called the **Cavalieri principle**. An example already due to Archimedes is the computation of the volume the half sphere of radius R, cut away a cone of height and radius R from a cylinder of height R and radius R. At height z, this body has a cross section with area $R^2\pi - r^2\pi$. If we cut the half sphere at height z, we obtain a disc of area $(R^2 - r^2)\pi$. Because these areas are the same, the volume of the half-sphere is the same as the cylinder minus the cone: $\pi R^3 - \pi R^3/3 = 2\pi R^3/3$ and the volume of the sphere is $4\pi R^3/3$. **Newton (1643-1727) and Leibniz(1646-1716)** developed calculus independently. It provided a new tool which made it possible to compute integrals through "anti-derivation". Suddenly, it became possible to find integrals using analytic tools. We can do this also in higher dimensions.

Examples

17.6. Find the volume of the unit sphere. **Solution:** The sphere is sandwiched between the graphs of two functions obtained by solving for z. Let R be the unit disc in the xy plane. If we use the **sandwich method**, we get

$$V = \int\int_R [\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} 1dz] dA \; .$$

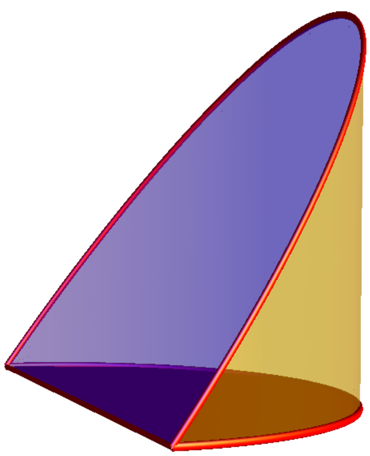
which gives a double integral $\int\int_R 2\sqrt{1-x^2-y^2} \; dA$ which is of course best solved in polar coordinates. We have $\int_0^{2\pi}\int_0^1 \sqrt{1-r^2} r \; drd\theta = 4\pi/3$.
With the **washer method** which is in this case also called **disc method**, we slice along the z axes and get a disc of radius $\sqrt{1-z^2}$ with area $\pi(1-z^2)$. This is a method suitable for single variable calculus because we get directly $\int_{-1}^1 \pi(1-z^2) \; dz = 4\pi/3$.

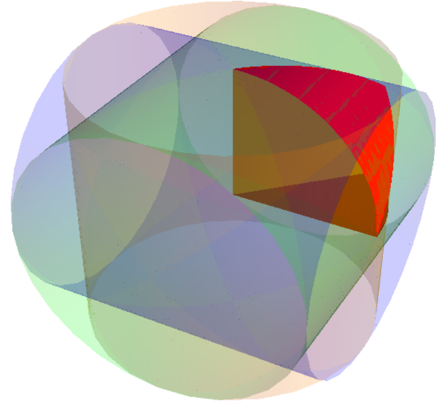
17.7. The mass of a body with mass density $\rho(x, y, z)$ is defined as $\int\int\int_R \rho(x, y, z)\, dV$. For bodies with constant density ρ, the mass is ρV, where V is the volume. Compute the mass of a body which is bounded by the parabolic cylinder $z = 4 - x^2$, and the planes $x = 0, y = 0, y = 6, z = 0$ if the density of the body is z. **Solution:**

$$\int_0^2 \int_0^6 \int_0^{4-x^2} z\, dz\, dy\, dx = \int_0^2 \int_0^6 (4 - x^2)^2/2\, dydx$$

$$= 6 \int_0^2 (4 - x^2)^2/2\, dx = 6(\frac{x^5}{5} - \frac{8x^3}{3} + 16x)|_0^2 = 2 \cdot 512/5$$

17.8. The solid region bound by $x^2 + y^2 = 1$, $x = z$ and $z = 0$ is called the **hoof of Archimedes**. It is historically significant because it is one of the first examples, on which Archimedes probed a Riemann sum integration technique. It appears in every calculus text book. Find the volume of the hoof. **Solution**. Look from the situation from above and picture it in the $x - y$ plane. You see a half disc R. It is the floor of the solid. The roof is the function $z = x$. We have to integrate $\int\int_R x\; dxdy$. We got a double integral problems which is best done in polar coordinates; $\int_{-\pi/2}^{\pi/2} \int_0^1 r^2 \cos(\theta)\; drd\theta = 2/3$.





17.9. Finding the volume of the solid region bound by the three cylinders $x^2 + y^2 = 1$, $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$ is one of the most famous volume integration problems going back to Archimedes.
Solution: look at 1/16'th of the body given in cylindrical coordinates $0 \le \theta \le \pi/4, r \le 1, z > 0$. The roof is $z = \sqrt{1 - x^2}$ because above the "one eighth disc" R only the cylinder $x^2 + z^2 = 1$ matters. The polar integration problem

$$16 \int_0^{\pi/4} \int_0^1 \sqrt{1 - r^2 \cos^2(\theta)} r\; drd\theta$$

has an inner r-integral of $(16/3)(1 - \sin(\theta)^3)/\cos^2(\theta)$. Integrating this over θ can be done by integrating $(1 + \sin(x)^3)\sec^2(x)$ by parts using $\tan'(x) = \sec^2(x)$ leading to the anti derivative $\cos(x) + \sec(x) + \tan(x)$. The result is $16 - 8\sqrt{2}$.

Homework

This homework is due on Tuesday, 7/30/2019.

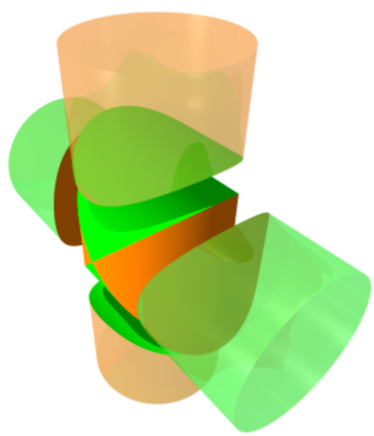
Problem 17.1: Evaluate the triple integral
$$\int_0^3 \int_0^z \int_0^{4y} 7e^{-y^2} z \, dxdydz \, .$$

Problem 17.2: Find the volume of the solid bounded by the paraboloids $z = x^2 + y^2$ and $z = 36 - (x^2 + y^2)$ and satisfying $x \geq 0, y \geq 0$.

Problem 17.3: Find the **moment of inertia** $\int\int\int_E (x^2 + y^2) \, dV$ of a cone
$$E = \{x^2 + y^2 \leq z^2 \; 0 \leq z \leq 15 \} \, ,$$
which has the z-axis as its center of symmetry.

Problem 17.4: Integrate $f(x, y, z) = x^2 + y^2 - z$ over the tetrahedron with vertices
$(0, 0, 0), (4, 4, 0), (0, 4, 0), (0, 0, 12)$.

Problem 17.5: This is a classic problem of Archimedes: What is the volume of the body obtained by intersecting the solid cylinders $x^2 + z^2 \leq 9$ and $y^2 + z^2 \leq 9$?



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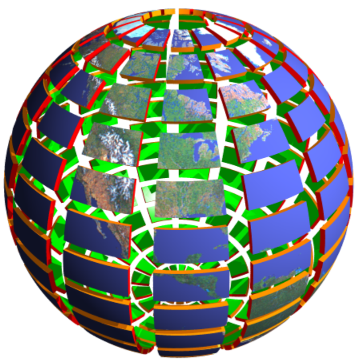
MATH S-21A

Unit 18: Spherical integrals

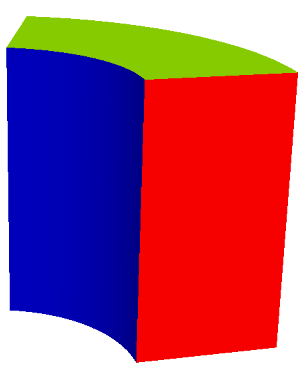
Lecture

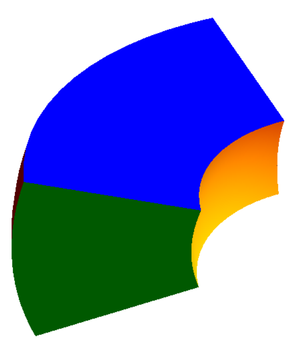
17.1. Cylindrical and spherical coordinate systems help to integrate in many situations.



Definition: Cylindrical coordinates are space coordinates where polar coordinates are used in the xy-plane and where the z-coordinate is untouched. The coordinate change transformation $T(r,\theta,z) = (r\cos(\theta), r\sin(\theta), z)$, produces the integration factor $\boxed{r}$. It is the same factor than the factor used in polar coordinates.

$$\iiint_{T(R)} f(x,y,z)\; dxdydz = \iiint_R g(r,\theta,z)\; \boxed{\mathrm{r}}\; drd\theta dz$$





Definition: **Spherical coordinates** use ρ, the distance to the origin as well as two **Euler angles**: $0 \leq \theta < 2\pi$ the polar angle and $0 \leq \phi \leq \pi$, the angle between the vector and the z axis. The coordinate change is

$$T : (x, y, z) = (\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi)) \ .$$

The integration factor measures the volume of a **spherical wedge** which is $d\rho, \rho\sin(\phi)\ d\theta, \rho d\phi = \rho^2\sin(\phi) d\theta d\phi d\rho$.

$$\int\int_{T(R)} f(x, y, z)\ dxdydz = \int\int_R g(\rho, \theta, z) \boxed{\rho^2\sin(\phi)}\ d\rho d\theta d\phi$$

A sphere of radius R has the volume

$$\int_0^R \int_0^{2\pi} \int_0^{\pi} \rho^2\sin(\phi)\ d\phi d\theta d\rho \ .$$

The most inner integral $\int_0^\pi \rho^2\sin(\phi) d\phi = -\rho^2\cos(\phi)|_0^\pi = 2\rho^2$. The next layer is, because ϕ does not appear: $\int_0^{2\pi} 2\rho^2\ d\phi = 4\pi\rho^2$. The final integral is $\int_0^R 4\pi\rho^2\ d\rho = 4\pi R^3/3$.

Definition: **The moment of inertia** of a body G with respect to an axis L is defined as the triple integral $\int\int\int_G r(x, y, z)^2\ dzdydx$, where $r(x, y, z) = \rho\sin(\phi)$ is the distance from the axis L.

Examples

17.2. For a sphere of radius R we obtain with respect to the z-axis:

$$I = \int_0^R \int_0^{2\pi} \int_0^{\pi} \rho^2\sin^2(\phi)\rho^2\sin(\phi)\ d\phi d\theta d\rho$$

$$= (\int_0^\pi \sin^3(\phi)\ d\phi)(\int_0^R \rho^4\ dr)(\int_0^{2\pi}\ d\theta)$$

$$= (\int_0^\pi \sin(\phi)(1 - \cos^2(\phi))\ d\phi)(\int_0^R \rho^4\ dr)(\int_0^{2pi}\ d\theta)$$

$$= (-\cos(\phi) + \cos(\phi)^3/3)|_0^\pi (L^5/5)(2\pi) = \frac{4}{3}\cdot\frac{R^5}{5}\cdot 2\pi = \frac{8\pi R^5}{15} \ .$$

17.3. If the sphere rotates with angular velocity ω, then $I\omega^2/2$ is the **kinetic energy** of that sphere. The moment of inertia of the earth for example is $8\cdot 10^{37} kgm^2$. The angular velocity is $\omega = 2\pi/day = 2\pi/(86400s)$. The rotational energy is $8\cdot 10^{37} kgm^2/(7464960000s^2) \sim 10^{29} J \sim 2.510^{24} kcal$.

17.4. Find the volume and the center of mass of a diamond, the intersection of the unit sphere with the cone given in cylindrical coordinates as $z = \sqrt{3}r$.
Solution: we use spherical coordinates to find the center of mass

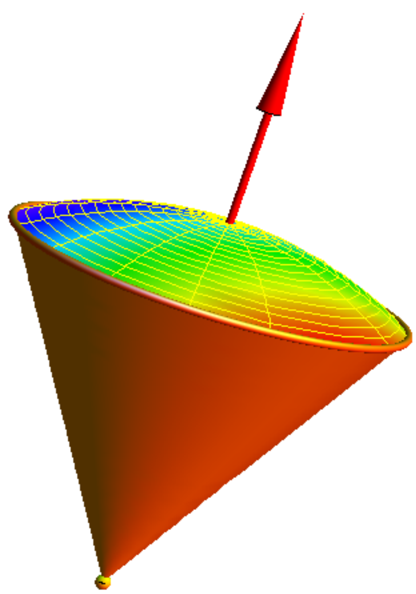
$$\begin{aligned}
\overline{x} &= \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^3 \sin^2(\phi)\cos(\theta)\; d\phi d\theta d\rho \frac{1}{V} = 0 \\
\overline{y} &= \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^3 \sin^2(\phi)\sin(\theta)\; d\phi d\theta d\rho \frac{1}{V} = 0 \\
\overline{z} &= \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^3 \cos(\phi)\sin(\phi)\; d\phi d\theta d\rho \frac{1}{V} = \frac{2\pi}{32V}
\end{aligned}$$

17.5. Find $\int\int\int_R z^2\; dV$ for the solid obtained by intersecting $\{1 \leq x^2 + y^2 + z^2 \leq 4\}$ with the double cone $\{z^2 \geq x^2 + y^2\}$.
Solution: since the result for the double cone is twice the result for the single cone, we work with the diamond shaped region R in $\{z > 0\}$ and multiply the result at the end with 2. In spherical coordinates, the solid R is given by $1 \leq \rho \leq 2$ and $0 \leq \phi \leq \pi/4$. With $z = \rho\cos(\phi)$, we have

$$\int_1^2 \int_0^{2\pi} \int_0^{\pi/4} \rho^4 \cos^2(\phi)\sin(\phi)\; d\phi d\theta d\rho$$

$$= (\frac{2^5}{5} - \frac{1^5}{5}) 2\pi (\frac{-\cos^3(\phi))}{3}|_0^{\pi/4} = 2\pi \frac{31}{5}(1 - 2^{-3/2})\ .$$

The result for the double cone is $\boxed{4\pi(31/5)(1 - 1/\sqrt{2}^3)}$.



Homework

This homework is due on Tuesday, 7/30/2019.

Problem 18.1: Assume the density of a solid $E = x^2 + y^2 - z^2 < 1, -1 < z < 1$ is given by the 10's power of the distance to the z-axes: $\sigma(x,y,z) = r^{10} = (x^2+y^2)^5$. Find its mass

$$M = \int \int \int_E (x^2+y^2)^5 \; dxdydz \; .$$

Problem 18.2: Find the moment of inertia $\int \int \int_E (x^2+y^2) \; dV$ of the body E whose volume is given by the integral

$$\text{Vol}(E) = \int_0^{\pi/4} \int_0^{\pi/2} \int_0^3 \rho^2 \sin(\phi) \; d\rho d\theta d\phi \; .$$

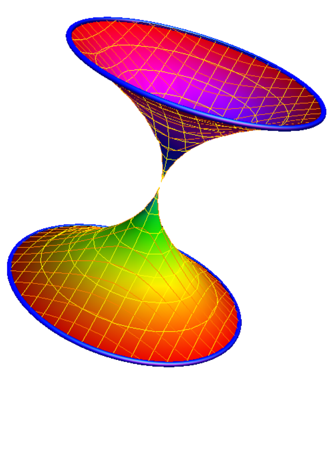
Problem 18.3: A solid is described in spherical coordinates by the inequality $\rho \leq 2\sin(\phi)$. Find its volume.

Problem 18.4: Integrate the function

$$f(x,y,z) = e^{(x^2+y^2+z^2)^{3/2}}$$

over the solid which lies between the spheres $x^2+y^2+z^2 = 1$ and $x^2 + y^2 + z^2 = 4$, which is in the first octant and which is above the cone $x^2+y^2 = z^2$.

Problem 18.5: Find the volume of the solid $x^2+y^2 \leq z^4, z^2 \leq 4$.



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MATH S-21A

Unit 19: Vectors fields

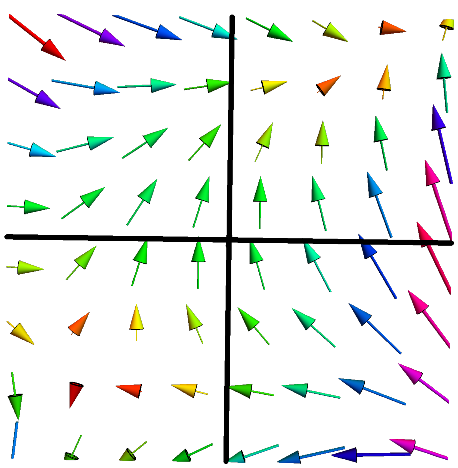
Lecture

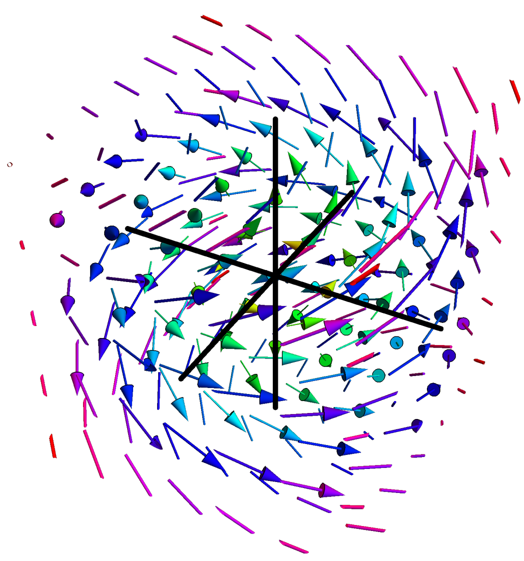
17.1. We have already seen geometries like curves, surfaces or solids. Then we have seen functions, which we consider to be scalar fields fields. If the function becomes vector valued we have **vector field**.

Definition: A **planar vector field** is a map F which assigns to a point $(x, y) \in \mathbb{R}^2$ a vector $\vec{F}(x, y) = [P(x, y), Q(x, y)]^T$. A **vector field in space** is a map, which assigns to each point $(x, y, z) \in \mathbb{R}^3$ a vector $\vec{F}(x, y, z) = [P(x, y, z), Q(x, y, z), R(x, y, z)]^T$.

17.2. Here are examples of vector fields in two and three dimensions

$$\vec{F}(x, y) = \begin{bmatrix} y - \sin(x) \\ x^3 + \cos(2y) \end{bmatrix}, \vec{F}(x, y, z) = \begin{bmatrix} -y \\ x \\ \sin(z) \end{bmatrix}.$$





Definition: If $f(x, y)$ is a function of two variables, then $\vec{F}(x, y) = \nabla f(x, y)$ is called a **gradient field**. Gradient fields in space are of the form $\vec{F}(x, y, z) = \nabla f(x, y, z)$. They are important!

17.3. When is a vector field a gradient field? $\vec{F}(x,y) = [P(x,y), Q(x,y)]^T = \nabla f(x,y)$ implies $Q_x(x,y) = P_y(x,y)$. If this does not hold at some point, $\vec{F}$ is no gradient field.

Clairaut test: If $Q_x(x,y) - P_y(x,y)$ is not zero at some point, then $\vec{F}(x,y) = [P(x,y), Q(x,y)]^T$ is not a gradient field.

17.4. We will see next week that $\text{curl}(\vec{F}) = Q_x - P_y = 0$ is also sufficient for $\vec{F}$ to be a gradient field if $\vec{F}$ is defined everywhere. How do we get f the function with $\vec{F} = \nabla f$? We will look at examples in class.

Examples

17.5. Is the vector field $\vec{F}(x,y) = [P,Q]^T = [3x^2y + y + 2, x^3 + x - 1]^T$ a gradient field? **Solution**: the Clairaut test shows $Q_x - P_y = 0$. We integrate the equation $f_x = P = 3x^2y + y + 2$ and get $f(x,y) = 2x + xy + x^3y + c(y)$. Now take the derivative of this with respect to y to get $x + x^3 + c'(y)$ and compare with $x^3 + x - 1$. We see $c'(y) = -1$ and so $c(y) = -y + c$. We see the solution $\boxed{x^3y + xy - y + 2x}$.

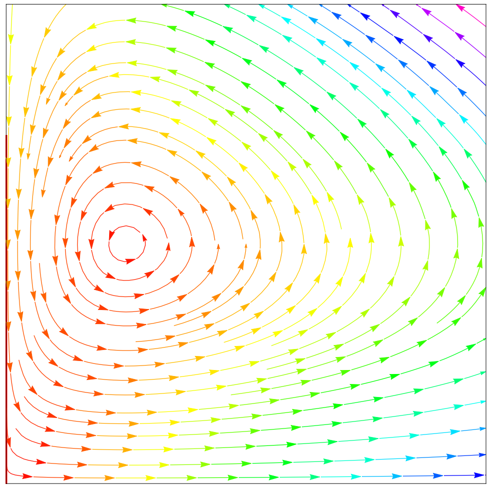
17.6. Is the vector field $\vec{F}(x,y) = [xy, 2xy^2]^T$ a gradient field? **Solution:** No: $Q_x - P_y = 2y^2 - x$ is not zero.

Vector fields appear naturally when studing differential equations. Here is an example in population dynamics:

17.7. If $x(t)$ is the population of a "prey species" like tuna fish and $y(t)$ is the population size of a "predator" like sharks. We have $x'(t) = ax(t) + bx(t)y(t)$ with positive a, b because both more predators and more prey species will lead to prey consumption. The rate of change of $y(t)$ is $y'(t) = -cy(t) + dxy$, where c, d are positive. This can be written using a vector field $\vec{r}' = \vec{F}(\vec{r}(t))$. We have a negative sign in the first part because predators would die out without food. The second term is explained because both more predators as well as more prey leads to a growth of predators through reproduction. A concrete example is the **Volterra-Lodka system**

$$\begin{aligned} \dot{x} &= 0.4x - 0.4xy \\ \dot{y} &= -0.1y + 0.2xy \ , \end{aligned}$$

where $\vec{F}(x,y) = [0.4x - 0.4xy$
$-0.1y + 0.2xy]^T$. Volterra explained with such systems the oscillation of fish populations in the Mediterranean sea. At any specific point $\vec{r}(x,y) = [x(t), y(t)]^T$, there is a curve $= \vec{r}(t) = [x(t), y(t)]^T$ through that point for which the tangent $\vec{r}\,'(t) = (x'(t), y'(t)$ is the vector field.

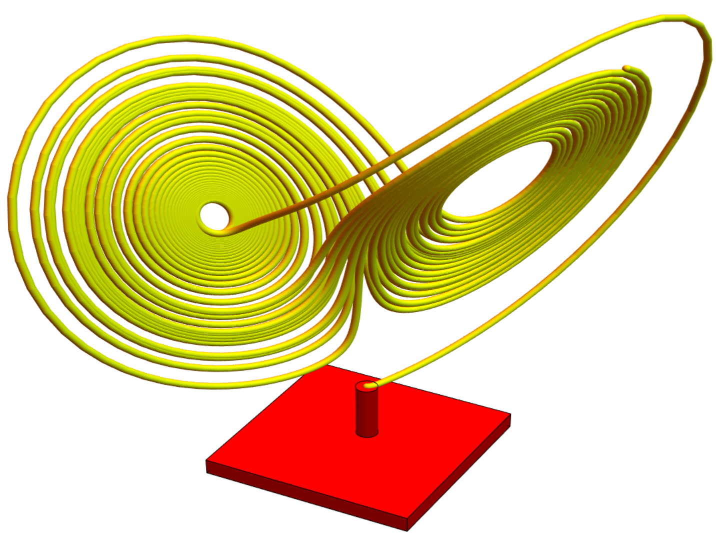


17.8. In mechanics the class of **Hamiltonian fields** plays an important role: if $H(x,y)$ is a function of two variables, then $[H_y(x,y), -H_x(x,y)]^T$ is called a **Hamiltonian vector field**. An example is the harmonic oscillator $H(x,y) = (x^2+y^2)/2$. Its vector field $(H_y(x,y), -H_x(x,y)) = (y,-x)$. The flow lines of a Hamiltonian vector fields are located on the level curves of H.

17.9. Here is a famous example. It is the **Lorenz vector field**

$$\vec{F}(x,y,z) = \begin{bmatrix} 10y - 10x \\ -xz + 28x - y \\ xy - \frac{8}{3}z \end{bmatrix} .$$

It features a so called **strange attractor**.



Homework

This homework is due on Tuesday, 7/30/2019.

Problem 19.1:
a) Draw the gradient vector field of $f(x,y) = \sin(x^2 - y^2)$.
b) Draw the gradient vector field of $f(x,y) = (x-1)^2 + (y-2)^2$.
In both cases, draw a contour map of f and use gradients to draw the vector field $F(x,y) = \nabla f$.

Problem 19.2: The vector field

$$\vec{F}(x,y) = \begin{bmatrix} \frac{x}{(x^2+y^2)^{(3/2)}} \\ \frac{y}{(x^2+y^2)^{(3/2)}} \end{bmatrix}$$

appears in electrostatics. Find a function $f(x,y)$ such that $\vec{F} = \nabla f$.

Problem 19.3:
a) Is the vector field $\vec{F}(x,y) = \begin{bmatrix} xy \\ x^2 \end{bmatrix}$ a gradient field?
b) Is the vector field $\vec{F}(x,y) = \begin{bmatrix} \sin(x) + y \\ \cos(y) + x \end{bmatrix}$ a gradient field?
In both cases, find $f(x,y)$ satisfying $\nabla f(x,y) = \vec{F}(x,y)$ or give a reason, why it does not exist.

Problem 19.4: Find conditions such that a vector field $\vec{F}(x,y,z)$ is a gradient field. Then check it in the following cases. If there is a gradient field, find f such that $\vec{F} = \nabla f$.
a) $\vec{F}(x,y,z) = [x^{11}, y^9, z]^T$.
b) $\vec{F}(x,y,z) = [y, x, z^3]^T$.
c) $\vec{F}(x,y,z) = [10y + 10x, 10x + 10y, x]^T$.
d) $\vec{F}(x,y) = [y, z, x]^T$.

Problem 19.5: Find the potential f to

$$\vec{F}(x,y,z) = [5x^4y + z^4 + y\cos(xy), x^5 + x\cos(xy), 4xz^3]^T \ .$$

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MULTIVARIABLE CALCULUS

MATH S-21A

Unit 20: Line integral theorem

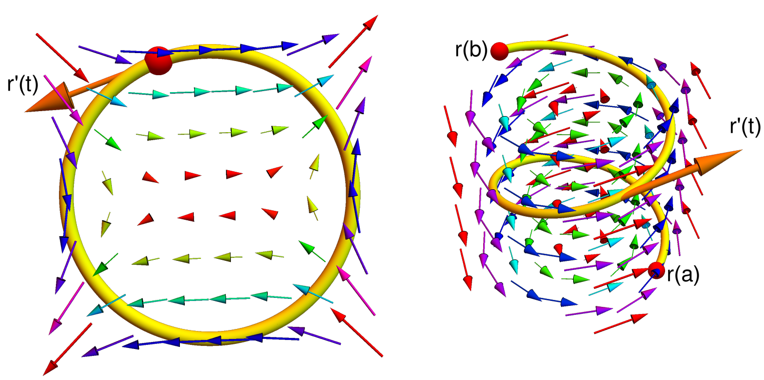
Lecture

17.1. Vector fields can be integrated along curves. If the vector field is a derivative, that is if it is a gradient field, then there is a fundamental theorem of line integrals which generalizes the fundamental theorem of calculus.

Definition: If $\vec{F}$ is a vector field in $\mathbb{R}^2$ or $\mathbb{R}^3$ and $C : t \mapsto \vec{r}(t)$ is a curve, then

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt$$

is called the **line integral** of $\vec{F}$ along the curve C.



17.2. We use also the short-hand notation $\int_C \vec{F} \cdot \vec{dr}$. In physics, if $\vec{F}(x,y,z)$ is a **force field**, then $\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)$ is called **power** and the line integral $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt$ is **work**. In electrodynamics, if $\vec{F}(x,y,z)$ is an electric field, then the line integral $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt$ is the **electric potential**.

17.3. Let $C : t \mapsto \vec{r}(t) = [\cos(t), \sin(t)]^T$ be a circle with parameter $t \in [0, 2\pi]$ and let $\vec{F}(x,y) = [-y, x]^T$. Calculate the line integral $I = \int_C \vec{F}(\vec{r}) \cdot \vec{dr}$.

Solution: We have $I = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt = \int_0^{2\pi} (-\sin(t), \cos(t)) \cdot (-\sin(t), \cos(t))\; dt = \int_0^{2\pi} \sin^2(t) + \cos^2(t)\; dt = 2\pi$

17.4. Let $\vec{r}(t)$ be a curve given in polar coordinates as $\vec{r}(t) = \cos(t), \phi(t) = t$ defined on $[0, \pi]$. Let $\vec{F}$ be the vector field $\vec{F}(x, y) = (-xy, 0)$. Calculate the line integral $\int_C \vec{F} \cdot \vec{dr}$.
Solution: In Cartesian coordinates, the curve is $r(t) = (\cos^2(t), \cos(t)\sin(t))$. The velocity vector is then $\vec{r}\,'(t) = [-2\sin(t)\cos(t), -\sin^2(t) + \cos^2(t)) = (x(t), y(t)]^T$. The line integral is

$$\int_0^\pi \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \int_0^\pi (\cos^3(t)\sin(t), 0) \cdot (-2\sin(t)\cos(t), -\sin^2(t) + \cos^2(t))\, dt$$

$$= \; -2\int_0^\pi \sin^2(t)\cos^4(t)\, dt = -2(t/16 + \sin(2t)/64 - \sin(4t)/64 - \sin(6t)/192)|_0^\pi = -\pi/8\,.$$

17.5. The first generalization of the fundamental theorem of calculus to higher dimensions is the **fundamental theorem of line integrals**.

Theorem: Fundamental theorem of line integrals: If $\vec{F} = \nabla f$, then

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = f(\vec{r}(b)) - f(\vec{r}(a))\,.$$

17.6. In other words, the line integral is the potential difference between the end points $\vec{r}(b)$ and $\vec{r}(a)$, if $\vec{F}$ is a gradient field.

Examples

17.7. Let $f(x, y, z)$ be the temperature distribution in a room and let $\vec{r}(t)$ the path of a fly in the room, then $f(\vec{r}(t))$ is the temperature, the fly experiences at the point $\vec{r}(t)$ at time t. The change of temperature for the fly is $\frac{d}{dt} f(\vec{r}(t))$. The line-integral of the temperature gradient ∇f along the path of the fly coincides with the temperature difference between the end point and initial point.

17.8. Here are some special cases: If $\vec{r}(t)$ is parallel to the level curve of f, then $d/dt f(\vec{r}(t)) = 0$ and $\vec{r}\,'(t)$ orthogonal to $\nabla f(\vec{r}(t))$. If $\vec{r}(t)$ is orthogonal to the level curve, then $|d/dt f(\vec{r}(t))| = |\nabla f||\vec{r}\,'(t)|$ and $\vec{r}\,'(t)$ is parallel to $\nabla f(\vec{r}(t))$.

17.9. The proof of the fundamental theorem uses the chain rule in the second equality and the fundamental theorem of calculus in the third equality of the following identities:

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)\, dt = \int_a^b \frac{d}{dt} f(\vec{r}(t))\, dt = f(\vec{r}(b)) - f(\vec{r}(a))\,.$$

Theorem: For a gradient field, the line-integral along any closed curve is zero.

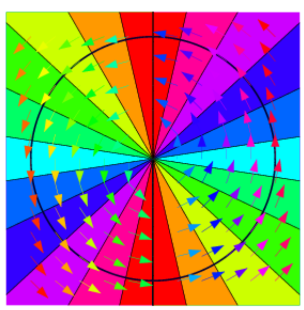
17.10. When is a vector field a gradient field? $\vec{F}(x,y) = \nabla f(x,y)$ implies $P_y(x,y) = Q_x(x,y)$. If this does not hold at some point, $\vec{F} = [P,Q]^T$ is no gradient field. This is called the **component test** or Clairaut test. We will see later that the condition $\text{curl}(\vec{F}) = Q_x - P_y = 0$ implies that the field is conservative, if the region satisfies a certain property.

17.11. Let $\vec{F}(x,y) = [2xy^2 + 3x^2, 2yx^2]^T$. Find a potential f of $\vec{F} = [P,Q]^T$.
Solution: The potential function $f(x,y)$ satisfies $f_x(x,y) = 2xy^2 + 3x^2$ and $f_y(x,y) = 2yx^2$. Integrating the second equation gives $f(x,y) = x^2y^2 + h(x)$. Partial differentiation with respect to x gives $f_x(x,y) = 2xy^2 + h'(x)$ which should be $2xy^2 + 3x^2$ so that we can take $h(x) = x^3$. The potential function is $f(x,y) = x^2y^2 + x^3$. Find g, h from $f(x,y) = \int_0^x P(x,y)\,dx + h(y)$ and $f_y(x,y) = g(x,y)$.

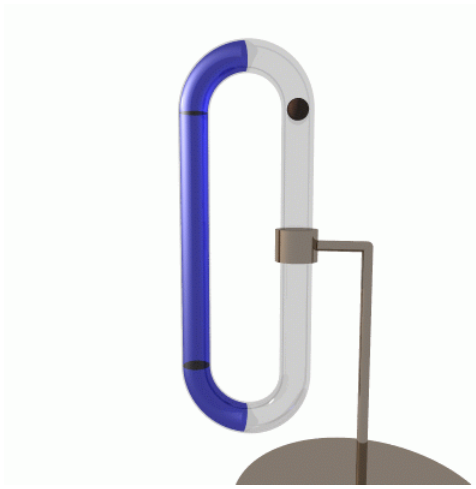
17.12. Let $\vec{F}(x,y) = [P,Q]^T = [\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}]^T$. It is a gradient field because $f(x,y) = \arctan(y/x)$ has the property that $f_x = (-y/x^2)/(1+y^2/x^2) = P, f_y = (1/x)/(1+y^2/x^2) = Q$. However, the line integral $\int_\gamma \vec{F}\,d\vec{r}$, where γ is the unit circle is

$$\int_0^{2\pi} [\frac{-\sin(t)}{\cos^2(t)+\sin^2(t)}, \frac{\cos(t)}{\cos^2(t)+\sin^2}]^T \cdot [-\sin(t), \cos(t)]^T\,dt$$

which is $\int_0^{2\pi} 1\,dt = 2\pi$. What is wrong?
Solution: note that the potential f as well as the vector-field F are not differentiable everywhere. The curl of F is zero except at $(0,0)$, where it is not defined.



17.13. A device which implements a non gradient force field is called a **perpetual motion machine**. It realizes a force field for which the energy gain is positive along some closed loop. The first law of thermodynamics forbids the existence of such a machine. It is informative to contemplate some of the ideas people have come up and to see why they don't work. Here is an example: consider a O-shaped pipe which is filled only on the right side with water. A wooden ball falls on the right hand side in the air and moves up in the water. You find plenty of other futile attempts on youtube.



Homework

This homework is due on Tuesday, 7/30/2019.

Problem 20.1: Let C be the space curve $\vec{r}(t) = [\cos(t), \sin(\sin(t)), t]^T$ for $t \in [0, \pi]$ and let $\vec{F}(x, y, z) = [y, x, 15]^T$. Find the value of the line integral $\int_C \vec{F} \cdot \vec{dr}$. You might want to use a theorem.

Problem 20.2: What is the work done by moving in the force field $\vec{F}(x, y) = [2x^3 + 1, 2y^4]^T$ along the parabola $y = x^2$ from $(-1, 1)$ to $(1, 1)$? a) compute it directly b) use the theorem.

Problem 20.3: Let $\vec{F}$ be the vector field $\vec{F}(x, y) = [-y, x]^T/2$. Compute the line integral of F along an ellipse $\vec{r}(t) = [a\cos(t), b\sin(t)]^T$ with width $2a$ and height $2b$. The result should depend on a and b.

Problem 20.4: It is hot. You unpack your portable swimming pool and place it in Harvard yard. Now, you swim along curve C given by part of the curve $x^{40} + y^{40} = 1$ in the first quadrant, oriented counter clockwise. There is a hose filling in fresh water to the tub so that there is a velocity field $\vec{F}(x, y) = [2x + 5y, 10y^4 + 5x]^T$ inside. Calculate the line integral $\int_C \vec{F} \cdot \vec{dr}$, the energy you gain from the fluid force when dislocating from $(1, 0)$ to $(0, 1)$. Be lazy.

Problem 20.5: Find a closed curve $C : \vec{r}(t)$ for which the vector field

$$\vec{F}(x, y) = [P(x, y), Q(x, y)]^T = [xy, x^2]^T$$

satisfies $\int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt \neq 0$.